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Scattering of Quantized Dirichlet Particles

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Abstract

As a step toward satisfactory understanding of the quantum dynamics of Dirichlet (D-) particles, the amplitude for the basic process describing the scattering of two quantized D-particles is computed in bosonic string theory. The calucluation is performed and cross-checked using three different methods, namely, (i) path integral, (ii) boundary state, and (iii) open-channel operator formalism. The analysis is exact in α' and includes the first order correction in the expansion with respect to the acceleration of the D-particles. The resultant Lorentz-invariant amplitude is capable of describing general non-forward scattering with recoil effects fully taken into account and it reproduces the known result for the special case of forward scattering in the limit of infinitely large D-particle mass. The expected form of the amplitude for the supersymmetric case is also briefly discussed.

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1 Introduction

A recent proposal on microscopic formulation of M theory [1, 2] suggests that the Dirichlet (D-)particles, which are thought to describe solitonic collective excitations of string[3], may actually play a much more fundamental role. In that approach, at least in the 11 dimensional infinite momentum frame, D-particles are the basic degrees of freedom and extended objects such as membranes, strings, etc., arise as collective excitations. A certain amount of evidence exists in support of this aspiring conjecture, essentially in low energy domain. It is extremely important to check if the details work out, especially whether there would be higher derivative corrections to the proposed action. To answer this question, obviously we need to know much more about the quantum interaction of D-particles.

The conjecture above was in part motivated by the effective low energy description of the D-brane interactions in the form of super-Yang-Mills theory on the worldvolume[4]. Although conceptually independent of M-theory, some hints about the hidden 11 dimensional feature have been uncovered[5][6][7], suggesting that extension of this approach is a viable road to understanding of the underlying fundamental theory. Another intriguing feature that appears in this approach is the apparent non-commutative nature of the coordinates describing the D-branes. Although natural from the gauge theory point of view, its physical meaning and how fundamental it actually is yet to be clarified. Again more detailed understanding of the quantum dynamics of D-branes is needed to make further progress.

With this background, we present in this article a computation of the amplitude for scattering of two quantized D-particles with finite mass in bosonic string theory. The result, valid when the accelerations of the D-particles are small in their mass scale, is nevertheless exact in α' expansion and describes general non-forward scattering with recoil effects fully included. In the special case of forward scattering in the limit of infinitely heavy classical D-particles, the amplitude correctly reduces to the known expression [8]. In fact, the structure of our result is formally quite similar to the one for the special case so that there exists a simple rule to go backwards, *i.e.* to go from this special case to our general case of quantum D-particles. Applying this rule, we will write down the result expected in the superstring case[1]as well in the final section. Our result should serve as a firm knowledge to be checked against in any proposal for the fundamental microscopic theory containing D-particles.

[1]In superstring case, satisfactory analysis should include the processes involving fermionic superpartners of (bosonic) D-particles and this is yet to be performed.

The actual calculations are perfomed and cross-checked by three different methods, namely (i) the path integral, (ii) the boundary state, and (iii) the operator method in the open string channel. The emphasis is on the path integral method, which is conceptually most complete in formulating the problem and hence is capable of computing the corrections in powers of acceleration systematically. It is a non-trivial extension of the formalism that we developed recently[9] for the scattering of closed string states from a quantized D-particle. We will see that all the details work out consistently , including the first order correction[2] that we will take into account in this article. As it will become clear later, the other two methods, although not suitable for computing corrections in powers of acceleration, have their own advantages. The boundary state method turned out to be most efficient for the purpose of computing the lowest order amplitude. On the other hand, the operator method reveals a deformation of the spectrum of an unusual type in the open string channel and the transitions among these excitations. Deeper understanding of this phenomena may be quite important in connection with the effective gauge theory.

In order to explain three different methods, each of which contains subtleties both technical and conceptual, in a reasonably self-contained manner, this article has become somewhat long. However, we have organized it in such a way that a hurried reader, if so desires, may consult section 2 on the path integral method, which by itself is complete, and skip sections 3 and 4 describing the other two methods. Here is the content of the rest of the article to follow:

Section 2. is devoted to the path integral method. In subsection 2.1, after describing how D0-D0 system should be characterized, we carefully analyze the boundary conditions when the D-particles follow arbitrary trajectories. They are then expressed in a suitable orthonormal coordinate system, to be used throughout the paper. Path integral over the string coordinate itself is performed in subsection 2.2. The complicated boundary conditions are implemented by devising a useful trick, and the lowest order amplitude as well as the Green's functions needed for the calculation of the corrections are computed. In subsection 2.3 the corrections, to first order in the acceleration, which arise from the extendedness of the boundaries are computed. The divergence produced in this process is consistently absorbed by the renormalization of the trajectories. Finally, in subsection 2.4 we quantize the D-particle trajectories themselves and obtain the desired quantum amplitude. It is then compared with the known result in the special limit already mentioned

[2]In this article, "order" invariably refers to the one in the power expansion with respect to the acceleration of the D-particles.

above.

In section 3, we briefly describe how the lowest order amplitude is reproduced using the boundary state representation of the interaction vertex developed previously[10][9]. After writing down the appropriate vertex states in 3.1, we sketch the actual computation in subsection 3.2. A powerful normal-ordering formula is developed (a proof is sketched in the Appendix) and used to calculate the Green's functions as well as the amplitude.

The computations performed in section 2 and 3 are essentially from the closed string channel. In section 4, we develop the operator formalism in the complimentary open string channel. First in 4.1, we make the modular transformation of the amplitude previously computed and obtain the form to be reproduced in the open channel. Upon performing the operator quantization in 4.2, we will find that the structure of the Hilbert space including the energy spectrum is rather unusual. We will give a careful analysis of this structure, give physical interpretation, and compute the relevant trace in a reliable way. The end result is exactly the one expected.

Finally, in section 5, we make several important remarks that emerge from our work. One among them is about the expected form of the amplitude in superstring theory. We will spell out how, by applying the aforementioned rule, the result of [8] for the special case can be promoted to the amplitude for quantized D-particles. Another remark concerns the existence of infinite number of diagrams contributing to the scattering in the high energy regime even for small string coupling.

2 Path Integral Approach

2.1 Characterization of D0-D0 system

The setup

A D-particle (D0-brane) is a point-like object which can emit and absorb a closed string. It does so in such a way that, when looked at from the (dual) open string channel, the ends of the open string lie somewhere on its worldline. Thus the simplest diagram describing two such D-particles interacting with each other looks like the one dipicted in Fig.1.

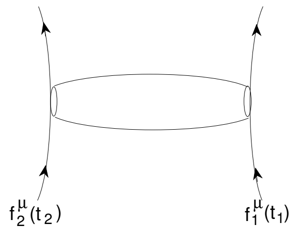

Fig.1 Basic process for D0-D0 scattering.

In the open string language, the relevant topology of the string worldsheet is an annulus Σ, which we take to be one with the inner radius R_1 and the outer radius R_2. The boundary circle at each end will be parametrized by the angle θ. We take the inner boundary to be mapped onto the worldline of a D-particle, parametrized by $f_1^\mu(t_1)$, while the outer boundary is embedded into the worldline of the other D-particle, described by $f_2^\mu(t_2)$. Until we come to the subsection 2.4, where we quantize the D-particles, we take $f_i^\mu(t_i)$, $i = 1, 2$ to be arbitrary yet fixed time-like trajectories[3] . As the ends of the open string may terminate anywhere on the worldlines, the Lorentz covariant constraints at the boundaries should be of the form [11]

$$X^\mu(r = R_i, \theta) \;\; = \;\; f_i^\mu(t_i(\theta))\,, \tag{2.1}$$

where $X^\mu(r, \theta)$ denote the open string variables in the worldsheet polar coordinates and $t_i(\theta)$ are arbitrary functions describing the embedding. This means that in the path integral formulation we develop in this section, we must integrate over $X^\mu(r, \theta)$ in the bulk and over $t_i(\theta)$ at the boundaries. Thus the relevant amplitude is

$$\mathcal{V}(f_1, f_2) \;\; = \;\; \int \mathcal{D}X^\mu(r, \theta) \prod_i \mathcal{D}t_i(\theta) \prod_i \delta(X^\mu(R_i, \theta) - f_i^\mu(t_i(\theta)) e^{-S[X]}\,, \tag{2.2}$$

where

$$S\,[X] \;\; = \;\; \frac{1}{4\pi\alpha'} \int_\Sigma d^2 z \partial_\alpha X^\mu \partial_\alpha X_\mu \tag{2.3}$$

is the open string action[4].

Scheme for $t_i(\theta)$-integration

To give a feeling for how we will compute the amplitude above, we need to briefly recall the scheme for $t_i(\theta)$-integration explained in [9] (for the single D-particle case). We first

[3]There will be a slight condition required on $f_i^\mu(t_i)$ for consistency later.

[4]We use Euclidean worldsheet and Minkowski target space with the metric $\eta_{\mu\nu} = \mathrm{diag}\,(-, +, +, \cdots, +)$.

split the integral over $t_i(\theta)$ into the one over the θ-independent mode, denoted by t_i, and the rest over the non-constant modes, and then, to retain general covariance, expand $f_i^\mu(t_i(\theta))$ in terms of the geodesic normal coordinate $\zeta_i(\theta)$ around $f_i^\mu(t_i)$. It can be written in the form

$$
\begin{aligned}
f_i^\mu(t_i(\theta)) &= f_i^\mu(t_i) + \dot{f}_i^\mu(t_i)\zeta_i(\theta) + \Omega_i^\mu(\theta)\,, && (2.4)\\
\text{where} \qquad \Omega_i^\mu(\theta) &\equiv \frac{1}{2}K_i^\mu \zeta_i(\theta)^2 \\
&\quad + \frac{1}{3!}\left(-\frac{\dot{f}_i^\mu}{h_i}K_i^2 - \frac{3}{2}\frac{\dot{h}_i}{h_i}K_i^\mu + P_i^{\mu\nu}\partial_t^3 f_{i\nu}\right)\zeta_i(\theta)^3 + \cdots\,. && (2.5)
\end{aligned}
$$

Here, a dot stands for a t-derivative and $h_i(t_i)$, $K_i^\mu(t_i)$ and $P_i^{\mu\nu}(t_i)$ are, respectively, the one-dimensional induced metric on the trajectory, the extrinsic curvature and a projection operator normal to the trajectory. They are given by

$$
\begin{aligned}
h_i &\equiv \dot{f}_i^\mu \dot{f}_{i\mu}\,, && (2.6)\\
K_i^\mu &\equiv \ddot{f}_i^\mu - \frac{1}{2}\frac{\dot{h}_i}{h_i}\dot{f}_i^\mu = P_i^{\mu\nu}\ddot{f}_{i\nu}\,, && (2.7)\\
P_i^{\mu\nu} &\equiv \eta^{\mu\nu} - h_i^{\mu\nu}\,, && (2.8)\\
h_i^{\mu\nu} &\equiv \frac{\dot{f}_i^\mu \dot{f}_i^\nu}{h_i}\,, && (2.9)
\end{aligned}
$$

where we have also introduced the projection operator $h_i^{\mu\nu}$ along the trajectory. The integration over $\zeta_i(\theta)$ cannot, unfortunately, be performed in an exact manner. As will be explained in detail in subsection 2.3, we will organize it perturbatively with respect to the order of t-derivatives. This expansion scheme is expected to be reliable when the accelerations of the D-particles are small compared with the D-particle mass scale. Diagramatically, it corresponds to the picture (see Fig.2) that the dominant interaction occurs at a point on the worldline and the corrections due to the extendedness of the boundaries are subsequently taken into account by $\zeta_i(\theta)$-integrations.

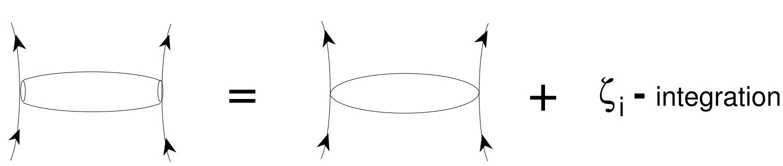

Fig.2 Diagram showing our approximation scheme.

Splitting of string coordinate

To perform the integration over $X^\mu(r,\theta)$, it is convenient to first split it into the θ-independent part $X_0^\mu(r)$ and the rest called $\tilde{\xi}^\mu(r,\theta)$:

$$\begin{aligned} X^\mu(r,\theta) &= X_0^\mu(r) + \tilde{\xi}^\mu(r,\theta)\,, && (2.10)\\ \int d\theta \tilde{\xi}^\mu(r,\theta) &= 0\,. && (2.11) \end{aligned}$$

Then the action separates without surface term into

$$S = \frac{1}{2\alpha'}\int \frac{dr}{r}(r\partial_r X_0(r))^2 + \frac{1}{4\pi\alpha'}\int d^2z(\partial_\alpha\tilde{\xi})^2\,. \tag{2.12}$$

The constraints at the boundaries now read

$$\begin{aligned} X_0^\mu(R_i) &= f^\mu(t_i) + \Omega^\mu_{i,0}\,, && (2.13)\\ \tilde{\xi}^\mu(R_i,\theta) &= \dot{f}_i^\mu(t_i)\zeta_i(\theta) + \tilde{\Omega}_i^\mu(\theta)\,, && (2.14)\\ \text{where} \qquad \Omega^\mu_{i,0} &\equiv \int\frac{d\theta}{2\pi}\Omega_i^\mu(\theta)\,, && (2.15)\\ \tilde{\Omega}_i^\mu(\theta) &\equiv \Omega_i^\mu(\theta) - \Omega^\mu_{i,0}\,. && (2.16) \end{aligned}$$

Note that by definition $\zeta_i(\theta)$ itself has no constant part, namely, $\int d\theta\zeta_i(\theta) = 0$.

Boundary conditions

We now formulate the boundary conditions that follow from the consistency under the variation of the action (2.12). As we have to take into account the constraints (2.13) and (2.14) already existing at the boundary, the analysis is somewhat involved. It is easy to see that the boundary conditions compatible with the equations of motion in the bulk are

$$\begin{aligned} \partial_r X_0(R_i)\cdot\delta X_0(R_i) &= 0\,, && (2.17)\\ \partial_r\tilde{\xi}(R_i,\theta)\cdot\delta\tilde{\xi}(R_i,\theta) &= 0\,. && (2.18) \end{aligned}$$

As has already been stated, we will treat $\mathcal{O}(\zeta^2)$ terms in (2.4) (which involve derivatives higher than the second) perturbatively. Therefore, these conditions need be imposed up to $O(\zeta)$. As for X_0^μ, the constraint (2.13) at the boundary says that the only possible variation up to this order is of the form $\delta X_0^\mu(R_i) = \dot{f}_i^\mu(t_i)\delta t_i$, *i.e.* in the direction of the trajectory. Similarly, (2.14) dictates that (since $\zeta_i(\theta)$ is considered small) possible variation of $\tilde{\xi}^\mu(R_i,\theta)$ is also along the trajectory. Thus, using the projection operators introduced in (2.8) and (2.9), we can write the conditions as

$$\begin{aligned} \delta X_0^\mu(R_i)P_{\mu\nu,i} &= 0\,, \qquad \partial_r X_0^\mu(R_i)h_{\mu\nu,i} = 0\,, && (2.19)\\ \delta\tilde{\xi}^\mu(R_i,\theta)P_{\mu\nu,i} &= 0\,, \qquad \partial_r\tilde{\xi}^\mu(R_i,\theta)h_{\mu\nu,i} = 0\,. && (2.20) \end{aligned}$$

To see the content of the conditions for $X_0^\mu(R_i)$ in more detail, it is useful to further split $X_0^\mu(r)$ into the classical part $X_{cl}^\mu(r)$ that satisfies $\partial^2 X_{cl}^\mu(r) = 0$, $X_{cl}^\mu(R_i) = f_i^\mu(t_i)$, and the quantum part. The desired splitting can be written as

$$X_0^\mu(r) = X_{cl}^\mu(r) + \frac{x_0^\mu(r)}{\sqrt{4\pi}}, \tag{2.21}$$

$$X_{cl}^\mu(r) = \frac{1}{s}\left(f_1^\mu(t_1)\ln\frac{b}{r} + f_2^\mu(t_2)\ln\frac{r}{a}\right), \tag{2.22}$$

$$\text{where} \qquad s \equiv \ln\frac{b}{a}. \tag{2.23}$$

Putting this into (2.19), we get

$$0 = \delta x_0^\mu(R_i) P_{\mu\nu,i}, \tag{2.24}$$

$$0 = \frac{1}{sR_i}(f_2(t_2) - f_1(t_1))^\mu h_{\mu\nu,i} + \frac{1}{\sqrt{4\pi}}\partial_r x_0^\mu(R_i) h_{\mu\nu,i}. \tag{2.25}$$

The first of these equations simply says that x_0^μ should satisfy the Dirichlet condition for the direction transverse to the trajectory at each end. The second, on the other hand, dictates that we should impose the Neumann condition for x_0^μ in the tangential direction and at the same time demand, for consistency,

$$(f_2(t_2) - f_1(t_1))_\mu \dot{f}_1^\mu(t_1) = (f_2(t_2) - f_1(t_1))_\mu \dot{f}_2^\mu(t_2) = 0. \tag{2.26}$$

These additional conditions may look odd at first sight, but their physical meaning is clear: The quantity $f_2^\mu(t_2) - f_1^\mu(t_1)$ is conjugate to the momentum transfer for the D-particles and the condition simply says that to $\mathcal{O}(\zeta)$ there should not be any momentum transfer along the trajectory so that the D-particles should be able to move freely along the trajectory[5]. We shall see later in sec. 3 that such conditions arise also from the requirement of BRST invariance for the boundary state representation of the interaction vertex.

The remaining analysis for $\tilde{\xi}^\mu$ is straightforward and we get the same type of boundary conditions as for x_0^μ. So if we define the total quantum fluctuation $\xi^\mu(r,\theta)$ by

$$\xi^\mu(r,\theta) = \frac{x_0^\mu(r)}{\sqrt{4\pi}} + \tilde{\xi}^\mu(r,\theta), \tag{2.27}$$

we can summarize the boundary conditions as

$$P_i^{\mu\nu}\xi_\nu(R_i,\theta) = 0, \tag{2.28}$$

$$h_i^{\mu\nu}\partial_r\xi_\nu(R_i,\theta) = 0. \tag{2.29}$$

[5]Remember that the effect of the extrinsic curvature starts at $\mathcal{O}(\zeta^2)$.

Orthonormal basis

The path integral over the quantum fluctuations $\xi^\mu(r,\theta)$ will be facilitated if we set up a suitable orthonormal basis. In the one particle case[9], there is a unique natural such frame, which is spanned by the unit vector $u^\mu = \dot{f}^\mu/\sqrt{-\dot{f}^2}$ tangent to the trajectory and the remaining $D-1$ transverse, mutually orthogonal unit vectors. Then the boundary conditions are neatly separated into the Neumann in the tangent and the Dirichlet in the transverse directions.

For the case at hand, however, we have *two ends* and the Neumann directions at these end points are *different*. Thus, we may consider two possible orthonormal frames, to be defined below.

Since $\dot{f}_i^\mu$ are expected to be time-like[6], let us first introduce the time-like unit vectors u_i defined by

$$u_i \equiv \frac{\dot{f}_i^\mu}{\sqrt{-h_i}}\,, \qquad u_i^2 = -1\,. \tag{2.30}$$

In the generic case, u_1 and u_2 are non-degenerate and define a plane, which we call "the trajectory plane". It is easy to show that the D-dimensional Lorentizan inner product $u_1 \cdot u_2$ takes the values in the range $\infty \geq -u_1 \cdot u_2 \geq 1$, where the equality on the right holds if and only if the their spatial parts agree, *i.e.* if $\vec{u}_1(t_1) = \vec{u}_2(t_2)$. Therefore we can parametrize $-u_1 \cdot u_2$ in the form

$$-u_1 \cdot u_2 = \cosh\chi\,, \tag{2.31}$$

where χ is real and positive (by convention) and it depends on $\dot{f}_i$ symmetrically. *This parameter χ will be of utmost importance as it will be shown, in subsection 2.4, to carry the essential information of the scattering.* Its physical meaning will also be spelled out there.

Now we can define the unit-normalized space-like vectors $\tilde{u}_2$ and $\tilde{u}_1$ in the trajectory plane, which are orthogonal to u_1 and u_2 respectively:

$$\tilde{u}_2 = \frac{1}{\sinh\chi}(u_2 - u_1\cosh\chi)\,, \qquad \tilde{u}_2^2 = 1\,, \qquad u_1\cdot\tilde{u}_2 = 0\,, \tag{2.32}$$

$$\tilde{u}_1 = \frac{1}{\sinh\chi}(u_1 - u_2\cosh\chi)\,, \qquad \tilde{u}_1^2 = 1\,, \qquad u_2\cdot\tilde{u}_1 = 0\,. \tag{2.33}$$

Either $\{u_1, \tilde{u}_2\}$ or $\{u_2, \tilde{u}_1\}$ will do, but we will choose the former for our basis in the trajectory plane. The latter then is expressed in terms of the former in the form

$$u_2 = u_1\cosh\chi + \tilde{u}_2\sinh\chi\,, \tag{2.34}$$

$$\tilde{u}_1 = -(u_1\sinh\chi + \tilde{u}_2\cosh\chi)\,. \tag{2.35}$$

[6]When the D-particles are put on-shell after quantization, this will be automatic.

By appending $D-2$ space-like orthonormal vectors u_I which span the space transverse to the trajectory plane, we complete our basis as

$$\{\hat{e}^\mu_A\} = \{u_1, \tilde{u}_2, u_I\}\ , \quad A=(i,I)\,, \quad i=1,2\,,\ I=3,4,\ldots,D\,. \tag{2.36}$$

Accordingly, we will define the components of ξ^μ in this frame as

$$\xi_A \equiv \hat{e}^\mu_A \xi_\mu\,, \tag{2.37}$$

that is, $\xi_1 \equiv u_1\cdot\xi$, $\xi_2 \equiv \tilde{u}_2\cdot\xi$ and $\xi_I \equiv u_I\cdot\xi$. With this notation, we have

$$a_\mu b^\mu = a_A b^A = a_A \eta^{AB} b_{BA} = a^A \eta_{AB} b^B\,, \tag{2.38}$$

$$\eta^{AB} = \eta_{AB} = (-,+,+,\cdots,+)\,, \tag{2.39}$$

for arbitrary Lorentz vectors a_μ, b_μ.

It is now a simple matter to express the boundary conditions (2.28) and (2.29) in our basis:

$$\partial_r\xi_1(R_1,\theta)=0\,, \tag{2.40}$$

$$\xi_2(R_1,\theta)=0\,, \tag{2.41}$$

$$\xi_1(R_2,\theta)\sinh\chi+\xi_2(R_2,\theta)\cosh\chi=0\,, \tag{2.42}$$

$$\partial_r\xi_1(R_2,\theta)\cosh\chi+\partial_r\xi_2(R_2,\theta)\sinh\chi\,, \tag{2.43}$$

$$\xi_I(R_1,\theta)=\xi_I(R_2,\theta)=0\,. \tag{2.44}$$

Note that these conditions are similar to but more involved than the ones that occur in the case of usual open string in a constant electromagnetic field [12, 13, 14] or in the case of D-particles moving in the same directions[8]. In the present case, the two ends couple to background trajectories which point in different directions and hence the boundary conditions for ξ_1 and ξ_2 cannot be fully disentangled.

2.2 Path integral over the string coordinates

Preliminary

We shall now perform the integration over $\xi^\mu(z)$. Taking into account the boundary constraints (2.13), (2.14), the action can be written as

$$S = S_{cl} + S_{cl-q} + S_q + S_{cst}\,, \tag{2.45}$$

$$S_{cl} = \frac{1}{2\alpha' s}(f_2-f_1)^2_\perp\,, \tag{2.46}$$

$$S_{cl-q} \;=\; \frac{1}{\alpha' s}(f_2 - f_1)^\mu_\perp (\Omega_{2,0} - \Omega_{1,0})_\mu \,, \tag{2.47}$$

$$S_q \;=\; \frac{1}{4\pi\alpha'}\int d^2 z (\partial_\alpha \xi)^2 \,, \tag{2.48}$$

$$S_{cst} \;=\; -i\int d\theta \sum_i \nu_\mu(R_i,\theta)\left(\tilde{\xi}^\mu(R_i,\theta) - \dot{f}^\mu_i(t_i)\zeta_i(\theta) - \tilde{\Omega}^\mu_i(\theta)\right) \,, \tag{2.49}$$

where $(f_2 - f_1)_\perp$ denotes the components orthogonal to the trajectory plane. In (2.49) we have introduced sources $\nu_\mu(R_i,\theta)$, *without constant parts*, at the boundaries in S_{cst}, which will be integrated to produce δ-functions enforcing the constraints (2.14). ((2.13) is already used in (2.47).)

The path integral over $\xi^\mu(z)$ then can be written in the form

$$\mathcal{V}_\xi \;=\; \int \mathcal{D}\xi^\mu e^{E_\xi} \,, \tag{2.50}$$

$$E_\xi \;=\; \frac{1}{4\pi\alpha'}\int d^2 z \xi(z)\cdot\partial^2\xi(z) + i\int d^2 z \xi(z)\cdot J(z) \,. \tag{2.51}$$

Although the source $J(z)$, representing $\nu_\mu(R_i,\theta)$, is active only on the boundaries, we shall, until appropriate time, take it to be a general function of z. This will allow us to compute the requisite Green's functions on the annulus satisfying the rather complicated boundary conditions imposed on $\xi(z)$. Also for a while we shall suppress the component indices for ξ and J unless needed.

In the following, it will be convenient to switch from the variable r to a more natural proper-time variable ρ defined by

$$\rho \;\equiv\; \ln\frac{r}{R_1} \,, \qquad 0 \le \rho \le s \equiv \ln\frac{R_2}{R_1} \,. \tag{2.52}$$

The Laplacian and the integration measure take the form, $\partial^2 = R_1^{-2} e^{-2\rho}(\partial_\rho^2 + \partial_\theta^2)$ and $\int d^2 z = R_1^2 \int_0^{2\pi} d\theta \int_0^s d\rho e^{2\rho}$.

We now expand various quantities in double Fourier series in θ and ρ and then integrate over the modes. First expand ξ and J in angular Fourier series:

$$\xi(\rho,\theta) \;=\; \sum_n \frac{1}{\sqrt{2}}(x_n(\rho) + i x'_n(\rho))\frac{e^{in\theta}}{\sqrt{2\pi}} \,, \tag{2.53}$$

$$J(\rho,\theta) \;=\; \sum_n \frac{1}{\sqrt{2}}(j_n(\rho) + i j'_n(\rho))\frac{e^{in\theta}}{\sqrt{2\pi}} \,. \tag{2.54}$$

From the reality conditions, x_n and x'_n must be real and satisfy $x_{-n} = x_n$, $x'_{-n} = -x'_n$. So the independent integration variables are x_n and x'_n for $n \ge 1$ and x_0. Then the exponent E_ξ becomes

$$E_\xi \;=\; -\frac{1}{4\pi\alpha'}\int_0^s d\rho \left[\sum_{n=0}^\infty x_n(-\partial_\rho^2 + n^2)x_n + \sum_{n\ge 1} x'_n(-\partial_\rho^2 + n^2)x'_n\right]$$

$$+iR_1^2 \int_0^s d\rho e^{2\rho} \left[\sum_{n=0}^{\infty} x_n j_n + \sum_{n\geq 1} x'_n j'_n \right] . \tag{2.55}$$

Since the primed system is identical to the unprimed (except for $n = 0$ part), we will hereafter exhibit the unprimed part only.

Next we wish to make a Fourier expansion of $x_n(\rho)$ with respect to ρ in the interval $[0, s]$. Here we must take into account the boundary conditions (2.40)∼(2.44), which in terms of $x_n(\rho)$'s read

$$\partial_\rho x_{1,n}(0) = 0\,, \tag{2.56}$$

$$x_{2,n}(0) = 0\,, \tag{2.57}$$

$$x_{1,n}(s) \sinh\chi + x_{2,n}(s) \cosh\chi = 0\,, \tag{2.58}$$

$$\partial_\rho x_{1,n}(s) \cosh\chi + \partial_\rho x_{2,n}(s) \sinh\chi = 0\,, \tag{2.59}$$

$$x_{I,n}(0) = x_{I,n}(s) = 0\,. \tag{2.60}$$

The expansion for $x_{I,n}$, which satisfies the usual Dirichlet condition at both ends, is standard. But finding the appropriate Fourier expansions for $x_{1,n}(\rho)$ and $x_{2,n}(\rho)$, obeying complicated boundary conditions, is rather difficult. To circumvent this, we will use the following trick. The idea is to first choose some appropriate complete basis, which satisfies a certain boundary condition, and then realize the actual boundary conditions (2.56) ∼ (2.60) by introducing extra source terms $\Delta j_n(\rho)$, to be discussed in detail later. For the moment, we include them in $j_n(\rho)$ and proceed.

To find a convenient as well as consistent Fourier basis, we must take into account that the conditions (2.56) ∼ (2.60) are not periodic in the interval $[0, s]$. This suggests that we should extend the functions into the enlarged interval $[-s, s]$ as *even* functions and then make the usual Fourier expansion. It reads

$$x_n(\rho) \;=\; \frac{1}{\sqrt{s}} \left(a_{n,0} + \sqrt{2} \sum_{m=1}^{\infty} a_{n,m} \cos k_m \rho \right) , \tag{2.61}$$

$$\text{where} \qquad k_m \;=\; \frac{\pi m}{s} \,. \tag{2.62}$$

Note that this automatically satisfies the *Neumann* condition at the ends of the *original* interval, *i.e.* at $\rho = 0, s$, which is still consistent with the true boundary conditions. We will use this as our base and append corrections to realize the true boundary conditions.

So we first compute the path integral for this basic case. Substituting (2.61) into (2.55), we get for the mode n (with the omission of the primed modes),

$$E_{\xi,n} \;=\; -\frac{n^2}{4\pi\alpha'} a_{n,0}^2 - \frac{1}{4\pi\alpha'} \sum_{m=1}^{\infty} (k_m^2 + n^2) a_{n,m}^2$$

$$+ia^2\left(a_{n,0}j_{n,0}+\sum_{m=1}^{\infty}a_{n,m}j_{n,m}\right)\,, \tag{2.63}$$

where

$$j_{n,0}=\frac{1}{\sqrt{s}}\int_0^s d\rho e^{2\rho}j_n(\rho)\,, \tag{2.64}$$

$$j_{n,m}=\sqrt{\frac{2}{s}}\int_0^s d\rho e^{2\rho}j_n(\rho)\cos k_m\rho\,. \tag{2.65}$$

The Gaussian integrations over the modes $a_{n,0}$ and $a_{n,m}$ are now trivial and it produces[7]

$$\int da_{n,0}\prod_{m=1}^{\infty}da_m\,e^{E_{\xi,n}}\quad\sim\quad D_n^{(N)}(s)\,e^{E(j)_n}\,, \tag{2.66}$$

where

$$\begin{aligned}
D_n^{(N)}(s) \;&\sim\; \frac{1}{n}\prod_{m=1}^{\infty}\left(k_m^2+n^2\right)^{-1/2}=\frac{1}{n}\left(\frac{2\sinh ns}{n}\right)^{-1/2}\,, &(2.67)\\
E(j)_n \;&=\; -\pi\alpha' R_1^4\left[\frac{1}{n^2}(j_{n,0})^2+\sum_{m=1}^{\infty}\frac{1}{k_m^2+n^2}(j_{n,m})^2\right]\\
&=\; -\pi\alpha' R_1^4\int_0^s d\rho\int_0^s d\rho' e^{2(\rho+\rho')}j_n(\rho)N_n(\rho,\rho')j_n(\rho')\,. &(2.68)
\end{aligned}$$

Here, $N_n(\rho,\rho')$ is the Neumann function for the operator $-\partial_\rho^2+n^2$ given by

$$\begin{aligned}
N_n(\rho,\rho') \;&=\; \frac{2}{s}\left(\frac{1}{2n^2}+\sum_{m=1}^{\infty}\frac{\cos k_m\rho\cos k_m\rho'}{k_m^2+n^2}\right)\\
&=\; \frac{1}{2n\sinh ns}\left(\cosh n(s-(\rho+\rho')+\cosh n(s-|\rho-\rho'|)\right) &(2.69)
\end{aligned}$$

with normalization $(-\partial_\rho^2+n^2)N_n(\rho,\rho')=\delta(\rho-\rho')$. These expressions are derived with the aid of the formulae

$$\begin{aligned}
\prod_{m=1}^{\infty}\left(1+\frac{y^2}{m^2}\right) \;&=\; \frac{\sinh\pi y}{\pi y}\,, &(2.70)\\
\sum_{m=1}^{\infty}\frac{\cos mx}{m^2+a^2} \;&=\; \frac{\pi}{2a}\frac{\cosh a(\pi-x)}{\sinh a\pi}-\frac{1}{2a^2}\,,\quad (0\le x\le 2\pi)\,. &(2.71)
\end{aligned}$$

We must now take the product over n. Due to the presence of the constant mode $a_{0,0}$, the contribution for $n=0$ is divergent. As we shall see shortly, for the actual boundary conditions for our problem, such divergences will cancel. Thus, for now we write $n=0$

[7] Here and hereafter, we will not be concerned with the overall constant. If desired, it can be most easily obtained by the open channel operator method, to be discussed in section 4.

as $n = n_0$, where it remains explicitly, and later check the cancellation. The front factor then becomes (including the contribution of $x'_n(\rho)$),

$$\begin{aligned} D^{(N)}(s) &= D_{n_0}(s) \prod_{n=1}^{\infty} D_n^{(N)}(s)^2 \\ &\sim \frac{s^{-1/2}}{n_0} \prod_{n=1}^{\infty} \left(\frac{\sinh ns}{n} \right)^{-1} \sim \frac{s^{-1/2}}{n_0} \eta(is/\pi)^{-1} , \end{aligned} \tag{2.72}$$

where we used a formula $\prod_{n=1}^{\infty} \sinh ns = \sqrt{2}\eta(is/\pi)$ yielding the Dedekind η-function $\eta(x) \equiv e^{i\pi x/12} \prod_{n=1}^{\infty}(1 - e^{2\pi i n x})$, and employed the standard ζ-function regularization, such as $\prod_{n=1}^{\infty} A = A^{-1/2}$ and $\prod_{n=1}^{\infty} n = \sqrt{2\pi}$, where appropriate.

Transverse sector

Let us now describe how one can realize the true boundary conditions (2.56) $\sim$ (2.60) by introducing extra source terms.

We begin with the case of transverse components $x_{I,n}(\rho)$, which satisfy the Dirichlet conditions at both ends, namely $x_{I,n}(0) = x_{I,n}(s) = 0$. In this case, we need not, of course, use any trick. Omitting the subscript I, we may simply expand $x_n(\rho)$ in the Fourier sine series

$$x_n(\rho) = \sqrt{\frac{2}{s}} \sum_{m=1}^{\infty} d_{n,m} \sin k_m \rho \, , \tag{2.73}$$

and go through the procedure similar to the previous case. One easily finds

$$D^{(D)}(s) \sim s^{-1/2} \eta(is/\pi)^{-1} \tag{2.74}$$

for the front determinant factor, and in the exponent of (2.68) the Neumann function is replaced by the Dirichlet function

$$D_n(\rho, \rho') = \frac{1}{2n \sinh ns} \left(-\cosh n(s - (\rho + \rho') + \cosh n(s - |\rho - \rho'|) \right) . \tag{2.75}$$

This suffices for the Dirichlet case. However, in preparation for handling the more difficult case of ξ_1-ξ_2 system, it is important to check if the same result can be reproduced by our more versatile trick, to be described below.

In terms of the modes defined in (2.61), the Dirichlet conditions can be implemented by the insertion of the following integrals representing δ-functions[8]:

$$\int d\mu_n \exp \left(\frac{i\mu_n}{\sqrt{s}} (a_{n,0} + \sqrt{2} \sum_{m=1}^{\infty} a_{n,m}) \right) , \tag{2.76}$$

[8] One might worry that imposition of the Dirichlet conditions on top of the Neumann conditions already built in is overconstraining. Classically, it is a legitimate concern. But in the path integral calculation, the effect of the extra Neumann conditions is to restrict the behavior of the functions only in the infinitesimal vicinity of the ends of the interval and this modification is of measure zero in the space of functions to be path-integrated.

$$\int d\mu'_n \exp\left(\frac{i\mu'_n}{\sqrt{s}}(a_{n,0} + \sqrt{2}\sum_{m=1}^{\infty} a_{n,m}(-1)^m)\right) . \tag{2.77}$$

The terms in the exponent can then be regarded as an addition of extra source terms of the form

$$\Delta j_{n,0} = \frac{1}{a^2\sqrt{s}}(\mu_n + \mu'_n) , \tag{2.78}$$

$$\Delta j_{n,m} = \frac{\sqrt{2}}{a^2\sqrt{s}}(\mu_n + (-1)^m\mu'_n) , \tag{2.79}$$

which, in the light of definitions (2.64), (2.65), can be written simply as

$$\Delta j_n(\rho) = \frac{2e^{-2\rho}}{R_1^2}(\mu_n\delta(\rho) + \mu'_n\delta(s-\rho)) . \tag{2.80}$$

In other words, we are putting random sources at the ends to enforce the Dirichlet conditions. Replacing $j_n(\rho)$ in (2.68) by $j_n(\rho) + \Delta j_n(\rho)$, we get, after some calculations, the following additional terms for the exponent:

$$\begin{aligned}\Delta E(j)_n &= = -4\pi\alpha'\left(N_n^0\tilde{\mu}_n^2 + \frac{1}{n^2N_n^0}\tilde{\mu'}_n^2\right) \\ &\quad +\pi\alpha' R_1^4\frac{N_n^0}{\sinh^2 ns}\left((\mathcal{J}_n(s))^2\tanh^2 ns + \left(\mathcal{J}_n(0) - \frac{1}{\cosh ns}\mathcal{J}_n(s)\right)^2\right) ,\end{aligned} \tag{2.81}$$

where

$$N_n^0 \equiv N_n(0,0) = N_n(s,s) = \frac{\cosh ns}{n\sinh ns} , \tag{2.82}$$

$$\mathcal{J}_n(s) \equiv \int_0^s d\rho e^{2\rho} j_n(\rho)\cosh n(s-\rho) , \tag{2.83}$$

$$\mathcal{J}_n(0) \equiv \int_0^s d\rho e^{2\rho} j_n(\rho)\cosh n\rho . \tag{2.84}$$

Modified sources $\tilde{\mu}_n$ and $\tilde{\mu'}_n$ are introduced in perfecting the squares but the Jacobian factor is unity for this rewriting. Upon integration over these sources, the determinant factors cancel, except for a factor of n. Taking the product over n, we then get $n_0(\prod_{n=1}^{\infty} n)^2 \sim n_0$, up to a constant. This extra factor of n_0 cancels $1/n_0$ in (2.72) and we reproduce (2.74). On the other hand, the remaining terms in (2.81) give, after some rearrangements,

$$-\pi\alpha' R_1^4\left(-\int_0^s d\rho\int_0^s d\rho' e^{2(\rho+\rho')} j_n(\rho) j_n(\rho')\frac{2}{2n\sinh ns}\cosh n(s-(\rho+\rho'))\right) , \tag{2.85}$$

which when added to the original piece (2.68) precisely effects the change $N_n(\rho,\rho') \to D_n(\rho,\rho')$. This demonstrates that our extra-source method indeed works nicely.

ξ_1-ξ_2 sector

With this warm up, let us now proceed to the main task of incorporating the boundary conditions for the ξ_1-ξ_2 system. Since the basic method has already been described in detail, we shall only give the essence.

The extra source terms for $x_{1,n}(\rho)$ and $x_{2,n}(\rho)$ to be added now take the form

$$\Delta j_n^1(\rho) = \frac{2e^{-2\rho}}{R_1^2}\mu'_n\delta(s-\rho)\sinh\chi\,, \tag{2.86}$$

$$\Delta j_n^2(\rho) = \frac{2e^{-2\rho}}{R_1^2}(\mu\delta(\rho)+\mu'_n\delta(s-\rho)\cosh\chi)\,. \tag{2.87}$$

Substituting the shifted expressions $j_n^i(\rho)+\Delta j_n^i(\rho)$ into the exponent (2.68), where now we must replace $j_n(\rho)N(\rho,\rho')j_n(\rho')$ by $j_n^i(\rho)\eta_{ij}N(\rho,\rho')j_n^j(\rho')$ with the metric $\eta_{ij}=(-,+)$, we get after some calculations

$$\begin{aligned} E_n(j) &= -4\pi\alpha'\left(N_n^0\tilde{\mu}_n^2+\frac{N_n^0H_n}{\cosh^2 ns}\tilde{\mu'}_n^2\right) \\ &\quad +\pi\alpha' R_1^4\Bigg[\frac{1}{n\sinh ns\cosh ns}\mathcal{J}_n^2(s)\mathcal{J}_n^2(s) \\ &\quad +\frac{\cosh ns}{nH_n\sinh ns}\left(\mathcal{J}_n^2(0)\cosh\chi-\mathcal{J}_n^1(0)\sinh\chi-\frac{\cosh\chi}{\cosh ns}\mathcal{J}_n^2(s)\right)^2\Bigg]\,, \end{aligned} \tag{2.88}$$

where the quantity H_n is defined by

$$H_n \equiv \sinh^2 ns-\sinh^2\chi=\frac{1}{4}e^{2ns}(1-e^{2\chi-2ns})(1-e^{-2\chi-2ns})\,, \tag{2.89}$$

and $\mathcal{J}_n^i(s), \mathcal{J}_n^i(0)$ are defined just as in (2.83) $\sim$ (2.84) with $j_n^i(\rho)$ in place of $j_n(\rho)$. Integrating over the sources $\tilde{\mu}_n, \tilde{\mu'}_n$, and forming the appropriate products over n (with the contribution of the primed system as well), the determinant factor becomes

$$\tilde{D}^{(12)}(s) \sim \frac{n_0^2 s}{\sinh\chi}\eta(is/\pi)^2 e^{2s/12}\prod_{n=1}^{\infty}(1-e^{2\chi-2ns})^{-1}(1-e^{-2\chi-2ns})^{-1}\,, \tag{2.90}$$

where the factor $n_0^2 s/\sinh\chi$ is due to the $n=0$ mode. This must be multiplied by the already existing factor $D^{(N)}(s)^2 \sim (1/n_0^2 s)\eta(is/\pi)^{-2}$ for two degrees of freedom. The factor $n_0^2 s$ and the η-functions cancel and we get

$$D^{(12)}(s) \sim \frac{1}{\sinh\chi}e^{2s/12}\prod_{n=1}^{\infty}(1-e^{2\chi-2ns})^{-1}(1-e^{-2\chi-2ns})^{-1}\,. \tag{2.91}$$

As for the Green's function, we can read off from (2.88) the corrections $\Delta G_{ij,n}$ to be added to $\eta_{ij}N_n(\rho,\rho')$. Omitting the details, the final result for the full Green's functions is

$$
\begin{aligned}
G_{11,n}(\rho,\rho') &= \frac{1}{2n}(\sinh n(\rho+\rho') + \sinh n|\rho-\rho'|) \\
&\quad -\frac{\sinh ns \cosh ns}{2nH_n}(\cosh n(\rho+\rho') + \cosh n(\rho-\rho'))\,, && (2.92) \\
G_{12,n}(\rho,\rho') &= \frac{\sinh 2\chi}{4nH_n}(\sinh n(\rho+\rho') - \sinh n(\rho-\rho'))\,, && (2.93) \\
G_{21,n}(\rho,\rho') &= \frac{\sinh 2\chi}{4nH_n}(\sinh n(\rho+\rho') + \sinh n(\rho-\rho'))\,, && (2.94) \\
G_{22,n}(\rho,\rho') &= \frac{1}{2n}(\sinh n(\rho+\rho') - \sinh n|\rho-\rho'|) \\
&\quad -\frac{\sinh ns \cosh ns}{2nH_n}(\cosh n(\rho+\rho') - \cosh n(\rho-\rho'))\,. && (2.95)
\end{aligned}
$$

One can check that they satisfy $(-\partial_\rho^2 + n^2)G_{ij,n}(\rho,\rho') = \eta_{ij}\delta(\rho-\rho')$ and that in the limit $\chi \to 0$ they reduce to the expected forms, namely, $G_{11,n} \to -N_n$, $G_{22,n} \to D_n$, and $G_{12,n}, G_{21,n} \to 0$. Moreover, they are of such forms that we can take the limit $n \to 0$ to obtain the Green's functions for the zero mode as well.

Summary of the lowest order amplitude

Let us now assemble the results so far obtained and write down the 0-th-order amplitude, *i.e.* yet without the corrections at the boundaries. It consists of the contribution from the classical part $e^{-S_{cl}}$ with S_{cl} given in (2.46), the one from each transverse coordinate ξ_I given in (2.74), and the one due to the ξ_1-ξ_2 system, (2.91). To these we must add the contribution of the b-c ghosts, which is , as usual, $D^{(gh)}(s) = \eta(is/\pi)^2$. Finally, we must integrate over the proper time s. Since we are dealing with a cylinder, the proper measure should simply be $\int_0^\infty ds$. Thus, using the explicit form of $\eta(is/\pi)$, we get

$$
\begin{aligned}
\mathcal{V}_0(f_1,f_2) &\sim \int_0^\infty ds e^{2s} e^{(D-26)s/12} s^{-(D-2)/2} e^{-(f_2-f_1)_\perp^2/2\alpha' s} \\
&\quad \cdot \frac{1}{\sinh\chi}\prod_{n=1}^\infty \left(1-e^{-2ns}\right)^{-(D-4)} \left(1-e^{-2ns+2\chi}\right)^{-1} \left(1-e^{-2ns-2\chi}\right)^{-1} && (2.96)
\end{aligned}
$$

as our lowest order approximation to the amplitude with fixed trajectories. In terms of the usual θ- and η-functions, it can be written compactly as

$$
\mathcal{V}_0(f_1,f_2) \sim \int_0^\infty ds s^{-(D-2)/2} e^{-(f_2-f_1)_\perp^2/2\alpha' s} \eta(\tau)^{-(D-2)} \frac{\theta_1'(0|\tau)}{\theta_1(\nu|\tau)}\,, \qquad (2.97)
$$

where $\tau = is/\pi$ and $\nu = -i\chi/\pi$.

2.3 Corrections at the boundaries

We shall now compute the corrections due to the extendedness of the boundaries, *i.e.* to the fluctuations of the geodesic coordinates $\zeta_i(\theta)$.

The first task is to derive the effective boundary action for $\zeta_i(\theta)$ arising from the integration over the special sources $\nu^\mu(R_i,\theta)$ that we introduced in (2.49) to enforce the constraints (2.14). These sources have until now been represented by the general source $J^A(z)$ (or its Fourier components j_n^A) and the integration over the string coordinates $\xi^\mu(z)$ produced an exponential factor with the exponent quadratic in J_n^A's connected by the Green's functions.

To apply this to the present case, we must recall the following: (i) $\nu^\mu(R_i,\theta)$'s do not have constant parts, since they are designed to couple only to the non-constant part of ξ_μ. (ii) Because, at each boundary, components of ξ_μ orthogonal to the direction of the trajectory vanish by the boundary conditions, the only components of $\nu^\mu(R_i,\theta)$ that actually couple to ξ_μ are the ones along the trajectory, namely $\nu(R_1,\theta)\cdot u_1$ and $\nu(R_2,\theta)\cdot u_2$. The point (i) says that we only need a Green's function in the space of non-zero modes (with respect to θ). Furthermore, the point (ii) tells us that only the Green's functions with components in the trajectory plane will be required.

Thus we form the following two-dimensional Green's function on the worldsheet without the zero mode:

$$G_{ij}(z,z') = G_{ij}(\rho,\theta;\rho',\theta') \equiv \sum_{n\geq 1} G_{ij,n}(\rho,\rho)\cos n(\theta-\theta')\,, \tag{2.98}$$

where $G_{ij,n}(\rho,\rho')$'s are as given in (2.92) $\sim$ (2.95). This satisfies $-\partial_z^2 G_{ij}(z,z') = \pi\tilde{\delta}^2(z,z')$, where $\tilde{\delta}^2(z,z') \equiv (1/R_1^2)e^{-2\rho}\delta(\rho-\rho')(\delta(\theta-\theta')-(1/2\pi))$ is the delta function in the space of non-zero modes. Then defining

$$\nu^i(\theta) \equiv \nu(R_i,\theta)\cdot u^i\,, \tag{2.99}$$

one finds that the exponential factor quadratic in the source takes the form

$$\exp\left(-\alpha'\int d\rho d\rho' d\theta d\theta' j^i(\rho,\theta)G_{ij}(\rho,\theta;\rho',\theta')j^j(\rho',\theta')\right)\,, \tag{2.100}$$

where

$$j^1(\rho,\theta) = \nu^1(\theta)\delta(\rho)+\nu^2(\theta)\delta(s-\rho)\cosh\chi\,, \tag{2.101}$$

$$j^2(\rho,\theta) = \nu^2(\theta)\delta(s-\rho)\sinh\chi\,. \tag{2.102}$$

Using the explicit forms of $G_{ij,n}(\rho,\rho')$, this exponential can be written as

$$\exp\left(-\alpha'\int d\theta d\theta' \nu^i(\theta)\mathcal{G}_{ij}(\theta,\theta')\nu^j(\theta')\right)\,, \tag{2.103}$$

with

$$\mathcal{G}_{ij}(\theta,\theta') = -\sum_{n=1}^{\infty}\frac{\sinh ns}{nH_n}\begin{pmatrix}\cosh ns & \cosh\chi \\ \cosh\chi & \cosh ns\end{pmatrix}_{ij}\cos n(\theta-\theta')\,. \tag{2.104}$$

Note that at this stage the Green's function, $\mathcal{G}_{ij}(\theta,\theta')$, has become completely symmetric in (i,j). This must be the case because the indices $i=1,2$ for $\nu^i(\theta)$ now refer to the two trajectories symmetrically.

We must further add to the exponent the remaining terms linear in the sources that directly couple to $\zeta_i(\theta)$'s (see (2.49)). It is convenient to separate out the parts containing $\nu^i(\theta)$ and write them in the form

$$i\int d\theta\left(\sum_i \nu^i(\theta)Z_i(\theta)-\Omega(\theta)\right)\,, \tag{2.105}$$

where

$$Z_i(\theta) \equiv \sqrt{-h_i}\,\zeta_i(\theta)+\frac{1}{3!}\frac{K_i^2}{\sqrt{-h_i}}\widetilde{\zeta_i(\theta)^3}+\cdots\,, \tag{2.106}$$

$$\Omega(\theta) \equiv \sum_i \nu_\mu(R_i,\theta)P_i^{\mu\nu}\left(\frac{1}{2}K_{i,\nu}\widetilde{\zeta_i(\theta)^2}+\cdots\right)\,. \tag{2.107}$$

$$\tag{2.108}$$

In the above, $\widetilde{\zeta_i(\theta)^n}$ means that the constant part contained in $\zeta_i(\theta)^n$ is removed. Then integration over $\nu^i(\theta)$ yields

$$(\det\mathcal{G})^{-1/2}\exp\left(\frac{1}{4\alpha'}\int d\theta d\theta' Z_i(\theta)\mathcal{G}^{-1}(\theta,\theta')^{ij}Z_j(\theta')\right)\,, \tag{2.109}$$

where the inverse Green's function $\mathcal{G}^{-1}(\theta,\theta')$ is easily obtained from (2.104) as

$$\mathcal{G}^{-1}(\theta,\theta') = -\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{\sinh ns}{nH_n}\begin{pmatrix}\cosh ns & -\cosh\chi \\ -\cosh\chi & \cosh ns\end{pmatrix}\cos n(\theta-\theta')\,. \tag{2.110}$$

We have now completed the calculation of the effective interaction of $\zeta_i(\theta)$'s. It consists of the factors $e^{-S_{cl-q}}$ from (2.47), $e^{-i\int d\theta\Omega(\theta)}$ from (2.105), and the factor (2.109) just computed. Note that *so far no approximation has been made.*

To proceed further, we now need to make an approximation. As was already announced, we will consider the case where the accelerations $\ddot{f}_i$ (and higher t_i-derivatives of $f_i^\mu(t_i)$) are small and treat the terms containing them as perturbations. Specifically, in this article we will keep only the terms linear in the extrinsic curvature $K^\mu = P^{\mu\nu}\ddot{f}_\nu$. Then, the expression we need to deal with is

$$\begin{aligned}
\mathcal{V}_{corr} &= (\det\mathcal{G})^{-1/2}\exp\left(\frac{1}{4\alpha'}\int d\theta d\theta'\sqrt{-h_i}\,\zeta_i(\theta)\mathcal{G}^{-1}(\theta,\theta')^{ij}\sqrt{-h_j}\,\zeta_j(\theta')\right) \\
&\quad\times\left[1-\frac{1}{2\alpha' s}(f_2-f_1)_{\perp\mu}\int\frac{d\theta}{2\pi}\left(K_2^\mu\zeta_2(\theta)^2-K_1^\mu\zeta_1(\theta)^2\right)\right. \\
&\quad\left.+\frac{1}{2}\sum_i\int d\theta\nu_\mu(R_i,\theta)P_i^{\mu\nu}K_{i,\nu}\widetilde{\zeta_i(\theta)^2}\right]. \qquad (2.111)
\end{aligned}$$

The Gaussian integration over $\zeta_i(\theta)$ yields the following effects. First, it produces the determinant factor, which consists of $(\det\mathcal{G})^{1/2}$ which cancels $(\det\mathcal{G})^{-1/2}$ in front and

$$(\prod_{n=1}(-h_1/\alpha')^{-1/2})^2(\prod_{n=1}(-h_2/\alpha')^{-1/2})^2 = \sqrt{-h_1/\alpha'}\sqrt{-h_2/\alpha'}\,, \qquad (2.112)$$

which is obtained by the Fourier mode expansion and the ζ-function regularization. The factor $\sqrt{-h_i}$ will be important in changing dt_i into the proper time $d\tau_i\sim dt\sqrt{-h_i}$. Second, the terms quadratic in $\zeta_i(\theta)$ are replaced by their quantum averages. The relevant correlation function $\langle\zeta_i(\theta)\zeta_j(\theta')\rangle$ is given by

$$\langle\zeta_i(\theta)\zeta_j(\theta')\rangle = -\frac{2\alpha'}{\sqrt{-h_i}\sqrt{-h_j}}\mathcal{G}_{ij}(\theta,\theta')\,. \qquad (2.113)$$

In particular, using the explicit form of the Green's function, the ones at the coincident point are computed as

$$\langle\zeta_i(\theta)^2\rangle = \frac{2\alpha'}{h_i}\mathcal{G}_{ii}(\theta,\theta)=\frac{2\alpha'}{h_i}\left(\ln\epsilon-\sum_{n=1}^\infty\frac{1}{nH_n}(\cosh 2\chi-e^{-2ns})\right)\,, \qquad (2.114)$$

where we have regularized the divergent sum $\sum_{n\ge1}(1/n)\to\sum_{n\ge1}(e^{-\epsilon n}/n)=-\ln\epsilon$. Since this is independent of θ, we immediately have $\langle\widetilde{\zeta_i(\theta)^2}\rangle=0$, so that the last term in (2.111) does not contribute to this order. As for the term containing f_2-f_1, we get a divergent term as well as a fininte correction. The former is of the form

$$-\frac{1}{s}(f_2-f_1)_\perp\cdot\left(\frac{K_2}{h_2}-\frac{K_1}{h_1}\right)\ln\epsilon\,. \qquad (2.115)$$

If we remember that we have, in the lowest order amplitude, the classical part of the form $\exp(-(1/2\alpha' s)(f_2-f_1)_\perp^2)$, we see that (2.115) can be absorbed by the renormalization of the trajectories (subscript R stands for renormalized quantities)

$$f_i^\mu = f_{i,R}^\mu+\delta f_{i,R}^\mu\,, \qquad (2.116)$$

$$\delta f_{i,R}^\mu = -\alpha'\frac{K_{i,R}^\mu}{h_i}\ln\epsilon\,. \qquad (2.117)$$

It is gratifying that (2.117) is exactly the form found for the one D-particle case treated in [9][9].

Putting all the findings together, we now have the final form of the amplitude for fixed trajectories, valid up to the first order in $\ddot{f}_i$:

$$\begin{aligned}\mathcal{V}(f_1,f_2) &= \int\prod_i dt_i\sqrt{-h_i/\alpha'}\int_0^\infty dse^{2s}e^{(D-26)s/12}s^{-(D-2/2)}e^{-(f_2-f_1)_\perp^2/2\alpha' s}\\ &\quad\cdot\frac{1}{\sinh\chi}\prod_{n=1}^\infty\left(1-e^{-2ns}\right)^{-(D-4)}\left(1-e^{-2ns+2\chi}\right)^{-1}\left(1-e^{-2ns-2\chi}\right)^{-1}\\ &\quad\cdot e^{(f_2-f_1)_\perp\cdot\Delta K}, \end{aligned} \tag{2.118}$$

$$\text{where}\qquad \Delta K^\mu \equiv \frac{1}{s}\left(\frac{K_2^\mu}{h_2}-\frac{K_1^\mu}{h_1}\right)\sum_{n=1}^\infty\frac{1}{nH_n}(\cosh 2\chi-e^{-2ns}). \tag{2.119}$$

2.4 Amplitude for quantized D-particles

Quantization of the collective coordinates

Having obtained the amplitude for fixed trajectories, we now quantize the trajectories themselves. The basic formalism developed in [9] works with a slight modification and hence we shall only describe the essence and refer the reader to [9] for further techinical details.

To the accuracy of our approximation, the action for a relativistic D-particle is of the form $S_0=-im_0\int_0^1 dt\sqrt{-\dot{f}^2})$ (with m_0 the bare mass), which, following [15], can be effectively turned into a more tractable quadratic action

$$S = \int_0^T d\tau\frac{1}{2}(\dot{f}^2(\tau)-m^2). \tag{2.120}$$

Here, m is the renormalized mass and $\tau(t)=(1/m)\int_0^t dt'\sqrt{-h(t')}$ is the (rescaled) proper time. Then the t-derivatives can be rewritten into τ-derivatives in the manner $\dot{f}(t)/\sqrt{-h(t)} = \dot{f}(\tau)/m$, $\ddot{f}(t)/(-h(t)) = \ddot{f}(\tau)/m^2$, etc. Hereafter, dots refer to τ-derivatives. The quantum amplitude is then obtained by $\mathcal{A}=\int_0^\infty dT\int\mathcal{D}f(\tau)\exp(iS)\mathcal{V}(f)$, where $\mathcal{V}(f)$ is the amplitude (or the "vertex operator" rather) for fixed trajectory.

For the present case, $\mathcal{V}(f)$ computed in the previous subsection depends on $\dot{f}_i$ and $\ddot{f}_i$ as well as on f_i. To separate out the dependence on the derivatives, it is convenient to replace them by new variables $v_{n,i}$, where $n(=1,2)$ refers to the number of derivatives,

[9] We have also confirmed that divergences that occur at the second order in K^μ are precisely absorbed by the renormalization of $\sqrt{-h_1}\sqrt{-h_2}$ with the same prescription (2.117). This is a highly non-trivial check of the consistency of our treatment.

via the insertion of unity:

$$1 \;=\; \int \prod_{n,i} dv_{n,i} \int \prod_{n,i} d\omega_{n,i} \exp\left(i \sum_{n,i} \omega_{n,i}(v_{n,i} - \partial^n f_i(\tau_i)) \right) . \tag{2.121}$$

Then the path integral over $f_i(\tau_i)$ we need to consider at the first stage is

$$\begin{aligned} \mathcal{A}_f \;\equiv\; & \int \prod_i \mathcal{D} f_i(\tau_i) \exp\left(i \sum_i \frac{1}{2} \int_0^{T_i} d\tau (\dot{f}_i^2(\tau) - m^2) \right) \\ & \cdot \int d\tau_1 d\tau_2 \exp\left(-(f_2(\tau_2) - f_1(\tau_1))_\perp^2/(2\alpha' s) + (f_2 - f_1)_\perp \cdot \Delta K(v_{n,i}) \right) \\ & \cdot \exp\left(-i \sum_{n,i} \omega_{n,i} \cdot \partial^n f_i(\tau_i)) \right) . \end{aligned} \tag{2.122}$$

Let us denote by f_i and f_i' the initial and the final positions of the D-particles respectively, and decompose $f_i^\mu(\tau_i)$ into the classical part $f_{cl,i}^\mu(\tau_i)$ and the quantum part $\tilde{f}_i^\mu(\tau_i)$:

$$f_i^\mu(\tau_i) \;=\; f_{cl,i}^\mu(\tau_i) + \tilde{f}_i^\mu(\tau_i) , \tag{2.123}$$

$$f_{cl,i}^\mu(\tau_i) \;=\; \frac{y_i^\mu}{T_i} \tau_i + f_i^\mu , \tag{2.124}$$

$$\text{where} \qquad y_i \;\equiv\; f_i' - f_i , \tag{2.125}$$

$$\tilde{f}_i(0) \;=\; \tilde{f}_i(T_i) = 0 . \tag{2.126}$$

Then due to the boundary condition (2.126) the classical and the quantum parts separate in the action:

$$\frac{1}{2} \int_0^{T_i} d\tau (\dot{f}_i^2 - m^2) \;=\; \frac{y_i^2}{2T_i} - \frac{1}{2} m^2 T_i + \frac{1}{2} \int_0^{T_i} d\tau \dot{\tilde{f}}_i^2 . \tag{2.127}$$

On the other hand, for the term $(f_2(\tau_2) - f_1(\tau_1))_\perp^2$ such a separation appears difficult at first sight. This problem,however, is solved by going to the momentum representation.

Let p_i and p_i' be the incoming and the outgoing momenta of the D-particles respectively as in Fig.3.

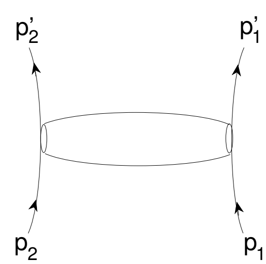

Fig.3 Scattering diagram in momentum space.

Then the factor to be multiplied for going to the momentum representation is

$$\begin{aligned}
&\exp(ip_1\cdot f_1 - ip_1'\cdot f_1')\exp(ip_2\cdot f_2 - ip_2'\cdot f_2') \\
&\quad = \exp(-i(y_1\cdot p_1' + y_2\cdot p_2'))\exp(i(f\cdot(p_1-p_1') + f_2\cdot(p_1+p_2-p_1'-p_2')))\,,
\end{aligned} \tag{2.128}$$

where we have defined $f\equiv f_1-f_2$. Integration over f_2 is now trivial and produces the momentum conserving δ-function $\delta(p_1+p_2-p_1'-p_2')$. As for the integration over f, it is of the form

$$I_f = \int d^D f e^{if\cdot(p_1-p_1')}e^{-(f+g)_\perp^2/(2\alpha' s)+(f+g)_\perp\cdot\Delta K}\,, \tag{2.129}$$

$$\text{where}\qquad g \equiv \frac{y_1}{T_1}\tau_1 - \frac{y_2}{T_2}\tau_2 + \tilde{f}_1(\tau_1) - \tilde{f}_2(\tau_2)\,. \tag{2.130}$$

So by shifting $f+g\to f$, we get

$$\begin{aligned}
I_f &= e^{-ig\cdot(p_1-p_1')}\int d^D f e^{if\cdot(p_1-p_1')}e^{-f_\perp^2/(2\alpha' s)+f_\perp\cdot\Delta K} \\
&\sim s^{(D-2)/2}\, e^{-ig\cdot(p_1-p_1')}e^{-(\alpha' s/2)(p_1-p_1'-i\Delta K)_\perp^2}\delta^2((p_1'-p_1)_\parallel)\,.
\end{aligned} \tag{2.131}$$

$\delta^2((p_1'-p_1)_\parallel)$ is the two-dimensional δ-function for the components in the plane spanned by the vectors $v_{i,1}$'s. Note that since g in the exponent is linear in y_i, the classical and the quantum parts are now separated. Note also that the front factor cancels $s^{-(D-2)/2}$ already present in the amplitude.

Next we perform the integration over y_i. Gathering together the terms containing y_i, the integral is

$$\begin{aligned}
I_y &= \int dy_1 dy_2 \exp\left\{i\sum_i\left(\frac{1}{2T_i}y_i^2 - \frac{\omega_{1,i}+(p_i-p_i')\tau_i}{T_i}\cdot y_i\right)\right\} \\
&\sim (T_1T_2)^{D/2}\exp\left(-\sum_i\frac{i}{2T_i}\left(\omega_{1,i}+p_i'T_i+\tau_i(p_i-p_i')\right)^2\right)\,,
\end{aligned} \tag{2.132}$$

where we have used momentum conservation.

Next we deal with the quantum fluctuations $\tilde{f}_i$. The integral to be performed is

$$I_{\tilde{f}} = \prod_{i=1}^{2}\int_{\tilde{f}_i(0)=\tilde{f}_i(T)=0}\mathcal{D}\tilde{f}_i\exp\left(\frac{i}{2}\int_0^T d\tau\dot{\tilde{f}}_i^2(\tau) - \sum_n i\omega_{n,i}\cdot\partial^n f_i(\tau_i) + i(p_i'-p_i)\cdot\tilde{f}_i(\tau_i)\right)\,. \tag{2.133}$$

This is precisely of the type that occured in [9] and can easily be evaluated by using the one-dimensional Green's function satisfying the Dirichlet conditions at the ends of the interval $[0,T]$. As it was shown there, with the standard ζ-function regularization, the

contributions of $\partial^n \tilde{f}(\tau_i)$ average out to zero for $n \geq 2$. Skipping the details, the result is (using again the momentum conservation)

$$I_{\tilde{f}} \;=\; (T_1 T_2)^{-D/2} \exp\left(\sum_i i\frac{\omega_{1,i}^2}{2T_i} - i\frac{(p'_i - p_i)\cdot\omega_{1,i}}{T_i}\left(\tau_i - \frac{T_i}{2}\right) + \frac{i}{2T_i}(p'_i - p_i)^2 \tau_i(\tau_i - T_i) \right) . \tag{2.134}$$

Combining (2.132) and (2.134), the front factors cancel and the exponent becomes

$$\sum_i \left(-\frac{i}{2}\omega_{1,i}(p_i + p'_i) + \frac{i}{2}(p'^2_i - p_i^2)\tau_i - \frac{i}{2}p'^2_i T_i \right) . \tag{2.135}$$

If we now put back the factors $\exp(i\sum_{n,i}\omega_{n,i}v_{n,i})$ (see (2.121)) and integrate over $\omega_{n,i}$, we get the following δ-functions:

$$\prod_i \delta(v_{1,i} - \frac{1}{2}(p_i + p'_i))\delta(v_{2,i}) . \tag{2.136}$$

This means that effectively we can replace $\dot{f}_i$ by $\frac{1}{2}(p_i + p'_i)$ and set $\ddot{f}_i$ to zero, exactly the same rule that we established in [9]. Therefore the correction ΔK that remained in (2.131) actually vanishes. Another consequence is that what has been referred to as the "trajectory plane" should now be understood as the one spanned by the mean momenta $\frac{1}{2}(p'_1 + p_1)$ and $\frac{1}{2}(p'_2 + p_2)$. Then when the D-particles are put on shell, we easily see that the momentum transfer $k = p'_1 - p_1 = -(p'_2 - p_2)$ is orthogonal to this plane since

$$k \cdot \frac{1}{2}(p'_1 + p_1) \;=\; \frac{1}{2}(p'^2_1 + m^2) - \frac{1}{2}(p_1^2 + m^2) = 0 , \tag{2.137}$$

$$-k \cdot \frac{1}{2}(p'_2 + p_2) \;=\; \frac{1}{2}(p'^2_2 + m^2) - \frac{1}{2}(p_2^2 + m^2) = 0 . \tag{2.138}$$

This is quite consistent with the presence of $\delta^2((p'_1 - p_1)_{\|})$ in (2.131) and allows us to replace $(p'_1 - p_1)^2_{\perp}$ in the exponent by $(p'_1 - p_1)^2$.

The remaining integrations over the interaction points τ_i and over T_i are trivial and they produce the four propagator legs[9], to be removed from the proper amplitude.

We may now write down the quantized proper amplitude in complete form. Omitting the momentum conserving δ-function and with the understanding that the D-particles are put on shell, it reads

$$\begin{aligned} \mathcal{A}_{bos}(p_1, p_2; p'_1, p'_2) \;\sim\; & \int_0^\infty ds e^{2s} e^{(D-26)s/12} e^{-(\alpha' s/2)(p'_1 - p_1)^2} \\ & \cdot\frac{1}{\sinh\chi}\prod_{n=1}^\infty \left(1 - e^{-2ns}\right)^{-(D-4)} \left(1 - e^{-2ns+2\chi}\right)^{-1} \left(1 - e^{-2ns-2\chi}\right)^{-1} , \end{aligned} \tag{2.139}$$

where χ is defined by

$$\cosh\chi = -u_1 \cdot u_2 \,, \tag{2.140}$$

$$u_i = \frac{p_i + p_i'}{\sqrt{-(p_i + p_i')^2}} \,. \tag{2.141}$$

This, however, is not quite the correct answer for fully quantum amplitude: We have not yet taken into account the quantum indistinguishability of D-particles. It is clearly seen when we express $\cosh\chi$ defined above in terms of the velocity v and the scattering angle θ_{CM} in the center of mass frame of the D-particles. It takes the form

$$\cosh\chi = \frac{1 + v^2\cos^2(\theta_{CM}/2)}{1 - v^2\cos^2(\theta_{CM}/2)} \,. \tag{2.142}$$

One sees that , for any v, χ vanishes for $\theta_{CM} = \pi$, *i.e.* for back-scattering but it does not vanish for the forward scattering. For indistinguishable particles this is obviously incorrect. The cure of course is to add the amplitude in which θ_{CM} is replaced by $\pi - \theta_{CM}$. This replacement must also be done for the exponential factor $\exp\left(-(\alpha' s/2)(p_1' - p_1)^2\right)$, which in the center of mass frame takes the form $\exp\left(-2\alpha' p^2 \sin^2(\theta_{CM}/2)\right)$, with p the maginitude of the spatial momentum. More generally the replacement to be made is $p_1' \leftrightarrow p_2'$, so that the desired quantum amplitude is

$$\mathcal{A}_{bos}(p_i, p_i') = \mathcal{A}_{bos}(p_1, p_2; p_1', p_2') + \mathcal{A}_{bos}(p_1, p_2; p_2', p_1') \,. \tag{2.143}$$

Forward scattering in the infinite mass limit

Let us check our result against the known expression[8, 16] in the special case of forward scattering for infinitely heavy D-particles. In this case, the particles are treated as distinguishable backgrounds and hence we should use (2.139) $\sim$ (2.141).

To make the comparison, we first turn the amplitude into the impact parameter representation by introducing a transverse vector b^μ and performing a Fourier transformation with respect to the momentum transfer $k_\perp = (p_1' - p_1)_\perp$. Since χ does not depend on the transverse components, this transformation simply produces the factor

$$\int d^{D-2}k_\perp e^{ik_\perp \cdot b} e^{-(\alpha' s/2)k_\perp^2} \sim s^{(D-2)/2} e^{-b^2/(2\alpha' s)} \,. \tag{2.144}$$

Let us now assume that the D-particles are moving in a common direction, say 1, with velocities V_1 and V_2. Then the 4-momenta of the particles are given by $p_i^\mu = (E_i, E_i V_i, \vec{0})$, where E_i is the energy of the particle #i, and they do not change as they scatter. Hence, the unit-normalized vectors u_i take a simple form

$$u_i = \frac{(1, V_i, \vec{0})}{\sqrt{1 - V_i^2}} \,. \tag{2.145}$$

From this $\sinh\chi$ is easily computed to be

$$\sinh\chi \quad = \quad \sqrt{(u_1\cdot u_2)^2-1} = \frac{|V_1-V_2|}{\sqrt{1-V_1^2}\sqrt{1-V_2^2}}\,, \tag{2.146}$$

which is exactly the expression that appears in [8, 16]. Applying (2.144) and (2.146) to (2.139), we recover the known result in this special case.

3 Boundary State Approach

In this section, we shall demonstrate that the lowest order amplitude for fixed trajectories obtained in (2.96) can be reproduced by using the boundary state representation developed in [10, 9]. To avoid possible confusion, we wish to emphasize at the outset that the "boundary state" that appears below *does not* correspond to a definite state of a D-particle. Rather it should properly be understood as a representation of the D0-D0-string interaction vertex describing the *transition* of a state of a D-particle into another state[10]. This feature explicitly appears as the presence of the derivative $\dot{f}_i$, which, when the D-particles are quantized as in the previous section, turns into $\frac{1}{2}(p_i+p_i')$ containing both the initial and the final momentum.

3.1 Boundary state representation of the vertex

The boundary ket $|B;f\rangle$ associated with a trajectory $f^\mu(t)$ is of the structure

$$|B;f\rangle \quad = \quad |B_z;f\rangle\otimes|B_{nz};f\rangle\otimes|B_{gh}\rangle\,, \tag{3.1}$$

where the subscripts "z", "nz" and "gh" stand for zero mode, non-zero modes and ghosts, respectively. The explicit forms for the non-zero modes and the ghost parts for our problem are given by

$$|B_{nz};f_1\rangle \quad = \quad \exp\left(-\sum_{n\ge1}\frac{1}{n}\alpha^\mu_{-n}D^\nu_\mu(\dot{f}_1)\tilde{\alpha}_{-n,\nu}\right)|0\rangle\,, \tag{3.2}$$

$$\langle B_{nz};f_2| \quad = \quad \langle 0|\exp\left(-\sum_{n\ge1}\frac{1}{n}\alpha^\mu_{n}D^\nu_\mu(\dot{f}_2)\tilde{\alpha}_{n,\nu}\right)\,, \tag{3.3}$$

$$|B_{gh}\rangle \quad = \quad \exp\left(\sum_{n=1}^{\infty}\left[\tilde{c}_{-n}b_{-n}+c_{-n}\tilde{b}_{-n}\right]\right)(c_0+\tilde{c}_0)|\downarrow\downarrow\rangle\,, \tag{3.4}$$

$$\langle B_{gh}| \quad = \quad \langle\uparrow\uparrow|(b_0-\tilde{b}_0)\exp\left(\sum_{n=1}^{\infty}\left[\tilde{c}_{n}b_{n}+c_{n}\tilde{b}_{n}\right]\right)\,, \tag{3.5}$$

$$\text{where}\qquad D^\nu_\mu(\dot{f}_i) \quad = \quad (h_i)_\mu^{\ \nu}-(P_i)_\mu^{\ \nu}\,. \tag{3.6}$$

[10]This feature is obscured in the case of infinitely heavy D-particles, since then their states do not change by the interaction and the "vertex" looks like an ordinary state.

The matrix $D^\nu_\mu(\dot{f}_i)$ has eigenvalue $+1$ along $\dot{f}_i$ (Neumann) direction and -1 for the transverse (Dirichlet) directions.

As for the zero-mode part $|B_z; f\rangle$, it should essentially be the position eigenstate $|f\rangle$ since the trajectory is given fixed. However, as was discussed in section 2.1, we must also incorporate the requirement that the momentum transfer in the tangential direction should vanish. This turned out to follow automatically from the requirement of BRST invariance. This invariance demands[12] $L_n|B_z; f\rangle \otimes |B_{nz}; f\rangle = \tilde{L}_{-n}|B_z; f\rangle \otimes |B_{nz}; f\rangle$, where L_n and $\tilde{L}_n$ are the closed string Virasoro generators in the usual notation. Then, since $\alpha^\mu_n|B_{nz}; f\rangle = -D^\mu_\nu(\dot{f})\tilde{\alpha}^\nu_{-n}|B_{nz}; f\rangle$ holds, one easily finds that we must have $p^\mu D^\nu_\mu(\dot{f}) = -p^\nu$, or equivalently, $p^\mu u_\mu = 0$ on $|B_z; f\rangle$, with u^μ the unit vector along $\dot{f}^\mu$. Thus the correct form of the zero-mode part should be, up to a constant,

$$|B_z; f_1\rangle = \delta(p \cdot u_1)|f_1\rangle \,, \tag{3.7}$$

$$\langle B_z; f_2| = \langle f_2|\delta(p \cdot u_2) \,. \tag{3.8}$$

One can check that in the case of infinitely heavy D-particles moving parallel to each other the ket (3.7) reduces to the one constructed in [16].

3.2 Calculation of the amplitude

The amplitude is now given by

$$\begin{aligned} \mathcal{V}_0(f_1, f_2) &= \langle B; f_2|\frac{1}{L_0 + \tilde{L}_0 - 2}U_0|B; f_1\rangle \\ &= \int_0^\infty ds \langle B; f_2|e^{-s(L_0+\tilde{L}_0-2)}U_0|B; f_1\rangle \,, \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} L_0 + \tilde{L}_0 - 2 &= \frac{\alpha'}{2}p^2 + \sum_{n=1}^\infty(\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n) \\ &\qquad + \sum_n n : (b_{-n}c_n + \tilde{b}_{-n}\tilde{c}_n) : -2 \end{aligned} \tag{3.10}$$

is the usual closed string Hamiltonian and U_0 is a ghost zero mode insertion, to be discussed below.

Let us now compute the contribution to the amplitude from each sector. First, for the zero mode sector, we have, ignoring the overall constant,

$$\begin{aligned} \mathcal{V}_z &= \langle f_2|\delta(p \cdot u_2)e^{(-\alpha' s/2)p^2}\delta(p \cdot u_1)|f_1\rangle \\ &\sim \int d^D p e^{ip\cdot(f_2-f_1)+(-\alpha' s/2)p^2}\delta(p \cdot u_1)\delta(p \cdot u_2) \\ &\sim \frac{s^{-(D-2)/2}}{\sinh\chi}e^{-(f_2-f_1)^2_\perp/(2\alpha' s)} \,. \end{aligned} \tag{3.11}$$

The exponent only contains $(f_2 - f_1)_\perp$, the components transverse to the trajectory plane, and this agrees with the discussion in section 2.1. The factor $1/\sinh\chi$ comes from the momentum integral in the trajectory plane, $\int d^2p\delta(p\cdot u_1)\delta(p\cdot u_2)$, which gives the inverse of the area spanned by the non-orthogonal unit vectors u_1 and u_2. The expression (3.11) agrees completely with the corresponding zero mode contribution computed by the path integral method.

Next, we turn to the non-zero modes. Propagation through the proper time s gives a factor e^{-2ns} inside the sum in the exponent of the boundary ket and hence what we need to compute is

$$
\begin{aligned}
\mathcal{V}_{nz} &= \langle 0|\exp\left(-\sum_{n\geq 1}\frac{1}{n}\alpha_n^\mu D_\mu^\nu(\dot{f}_2)\tilde{\alpha}_{n,\nu}\right) \\
&\qquad \cdot\exp\left(-\sum_{n\geq 1}\frac{1}{n}e^{-2ns}\alpha_{-n}^\mu D_\mu^\nu(\dot{f}_1)\tilde{\alpha}_{-n,\nu}\right)|0\rangle\,. \qquad (3.12)
\end{aligned}
$$

This can be evaluated by normal ordering of the oscillators. In the Appendix, we derive a complete normal-ordering formula[11] by extending the method developed in [17]. Let $(a_i, a_i^\dagger)$ and $(\tilde{a}_i, \tilde{a}_i^\dagger)$ be two independet sets of oscillators satisfying the usual (anti-)commutation relations $\left[a_i, a_j^\dagger\right]_\pm = \left[\tilde{a}_i, \tilde{a}_j^\dagger\right]_\pm = \delta_{ij}$, A and B be arbitrary matrices, and write $aA\tilde{a} \equiv \sum_{i,j} a_i A_{ij}\tilde{a}_j$ etc.. Then we have

$$
\begin{aligned}
&\exp\left(aA\tilde{a}\right)\exp\left(\tilde{a}^\dagger B a^\dagger\right) \\
&= \left[\det(1-\epsilon BA)\right]^{-\epsilon}\exp\left(\tilde{a}^\dagger(1-\epsilon BA)^{-1}Ba^\dagger\right)\exp\left(aA(1-\epsilon BA)\tilde{a}\right) \\
&\quad \cdot\exp\left(-a^\dagger\ln(1-\epsilon B^T A^T)a - \tilde{a}^\dagger\ln(1-\epsilon BA)\tilde{a}\right)\,, \qquad (3.13)
\end{aligned}
$$

where $\epsilon = +1\,(-1)$ for commuting (anti-commuting) oscillators. Setting

$$
a_{n,\mu} = \frac{i}{\sqrt{n}}\alpha_{n,\mu}\,, \qquad a_{n,\mu}^\dagger = \frac{1}{i\sqrt{n}}\alpha_{-n,\mu}\,, \qquad (3.14)
$$

$$
A_{n\mu,\ell\nu} = D_{\mu\nu}(f_2)\delta_{n\ell}\,, \qquad B_{n\mu,\ell\nu} = e^{-2ns}D_{\mu\nu}(f_1)\delta_{n\ell}\,, \qquad (3.15)
$$

we readily obtain

$$
\mathcal{V}_{nz} = \prod_{n=1}^{\infty}\left[\det\left(1 - D_1 D_2 e^{-2ns}\right)\right]^{-1}\,, \qquad (3.16)
$$

[11] For the present purpose, we only need the formula for the vacuum expectation value, which is certainly well-known. The full formula given below, however, is needed in more involved computations (such as those of Green's functions, etc.). Since, to our knowledge, it has not been recorded in the literature, we provide a derivation in the Appendix.

where the determinant here is over the space-time indices only and we have written D_i for $D^\nu_\mu(\dot{f}_i)$. It is easy to show that in our orthonormal basis $\{u_1, \tilde{u}_2, u_I\}$ defined in (2.36), the D_i's have the following representation:

$$\begin{aligned} D_i &= d_i \oplus (-\mathbf{1}_{D-2}) \,, &(3.17)\\ d_1 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , &(3.18)\\ d_2 &= \begin{pmatrix} \cosh 2\chi & \sinh 2\chi \\ -\sinh 2\chi & -\cosh 2\chi \end{pmatrix} , &(3.19)\end{aligned}$$

where $\mathbf{1}_{D-2}$ stands for the unit matrix in the $D-2$ dimensional space spanned by u_I. Therefore

$$D_1 D_2 = d_1 d_2 \oplus \mathbf{1}_{D-2} = \begin{pmatrix} \cosh 2\chi & \sinh 2\chi \\ \sinh 2\chi & \cosh 2\chi \end{pmatrix} \oplus \mathbf{1}_{D-2} \,. \qquad (3.20)$$

Note that the matrix $d_1 d_2$ is of the form of a Lorentz boost in two dimensions and hence its eigenvalues are $e^{\pm 2\chi}$. Therefore we get

$$\begin{aligned} \mathcal{V}_{nz} &= \prod_{n=1}^{\infty} \left[\det \left(1 - D_1 D_2 e^{-2ns} \right) \right]^{-1} \\ &= \prod_{n=1}^{\infty} \left(1 - e^{-2ns+2\chi} \right) \left(1 - e^{-2ns-2\chi} \right) \left(1 - e^{-2ns} \right)^{-(D-2)} . \qquad (3.21)\end{aligned}$$

Finally, consider the ghost contribution. As is well-known, due to the existance on the cylinder of one Teichmüller parameter and one conformal symmetry (rotation), we need to insert a zero mode $U_0 = (c_0 - \tilde{c}_0)(b_0 + \tilde{b})$ in order to get a non-vanishing inner product[12]. Then, applying (3.13) with $\epsilon = -1$, we get the familiar answer

$$\mathcal{V}_{gh} = \langle B_{gh} | e^{-s(L_0^{gh} + \tilde{L}_0^{gh})} U_0 | B_{gh} \rangle = \prod_{n=1}^{\infty} (1 - e^{-2ns})^2 \,. \qquad (3.22)$$

Assembling altogether we thus obtain

$$\begin{aligned} \mathcal{V}_0(f_1, f_2) \sim & \int_0^\infty ds e^{2s} s^{-(D-2)/2} e^{-(f_1(t_1) - f_2(t_2))^2_\perp / 2\alpha' s} \\ & \cdot \frac{1}{\sinh \chi} \prod_{n=1}^{\infty} (1 - e^{-2ns})^{-(D-4)} \left(1 - e^{-2ns+2\chi} \right)^{-1} \left(1 - e^{-2ns-2\chi} \right)^{-1} \qquad (3.23)\end{aligned}$$

which, for $D = 26$, is identical to the lowest order result (2.96) obtained by the path integral method.

We can also compute the Green's function $G_{ij}(z, z')$ (see (2.98)), satisfying the rather complicated boundary conditions, by using the boundary state representation. Since the

one we need is in the space of non-zero modes, the relevant quantity is the non-zero mode part of the amplitude with two closed string tachyon vertices inserted

$$\mathcal{A}_{nz}(k,k') \;=\; \langle B_{nz}; f_2| e^{ik\cdot X^{(+)}(z)} e^{ik\cdot X^{(-)}(z)} e^{ik'\cdot X^{(+)}(z')} e^{ik'\cdot X^{(-)}(z')} |B_{nz}; f_1\rangle\,, \qquad (3.24)$$

where $X^{(\pm)}$ are the non-zero mode parts of the closed string cooridnate given by $X^{(\pm)}(z) = \mp i\sum_{n=1}^{\infty}(1/n)(\alpha_{\mp n}z^{\pm n} + \tilde{\alpha}_{\mp n}\bar{z}^{\pm n})$. To compute this amplitude, the normal-ordering formula (3.13) is indispensable. As we wish to check only the non-trivial components $G_{ij}(z,z')$ with i,j in the trajectory plane, we restrict the momenta k and k' to have components only in this plane. Then, combining with appropriate use of coherent state method, we obtain, after a long calculation,

$$\mathcal{A}_{nz}(k,k') \;=\; D\exp\left(4k^i G_{ij}(z,z')k'^j\right)\,, \qquad (3.25)$$

where D is the determinant factor (3.16), k^i stands for $k\cdot u^i$ etc., and $G_{ij}(z,z')$ is precisely the one that we computed in the previous section by the path integral method.

4 Operator Formalism in Open String Channel

4.1 Amplitude in the open string channel

In the previous two sections, we have computed the D0-D0 scattering amplitude essentially from the closed string channel. An important question is to see how, in the low energy domain, the result can be understood in terms of a spontaneously broken effective gauge theory[4], which is based on the open channel picture. Although there is no doubt that this gauge theory description is basically correct and is extremely useful, how it should work for D-particles with finite mass (and hence with recoil) is still a non-trivial issue. As was already emphasized in the introduction, it must be intimately related with the "non-commutative nature of spacetime" and clarifying this connection would help understand the physical meaning of this intriguing concept. This makes the following re-derivation of the amplitude in the open channel a significant exercise.

Before taking up this task, let us first perform the modular transformation of the lowest order amplitude (2.97) to see what to expect. Introducing the usual open channel modular parameter w as

$$w \;\equiv\; e^{2\pi^2/s}\,, \qquad \frac{-\ln q}{\pi} = \frac{-2\pi}{\ln w}\,, \qquad (4.1)$$

and using the standard formulas for the modular transformation of the θ-functions, the expression (2.97) for $D = 26$ is transformed into

$$\mathcal{V}_0(f_1, f_2) \sim \int_0^1 \frac{dw}{w^2} w^{(f_1-f_2)_\perp^2/(4\pi^2\alpha')} w^{\nu(1-\nu)/2} f(w)^{-24} \times \frac{1}{1-w^\nu} \left[\frac{f(w)^2}{\prod_{m=1}^\infty (1-w^{m+\nu})(1-w^{m-\nu})} \right] . \quad (4.2)$$

where

$$\nu \equiv -i\frac{\chi}{\pi} , \qquad f(w) \equiv \prod_{m=1}^\infty (1-w^m) . \quad (4.3)$$

This indicates that apparently the open string spectrum is a peculiar one in that (i) the Casimir energy $\nu(1-\nu)/2$ is complex, (ii) two types of non-zero modes exist with complex energy levels $m+\nu$ and $m-\nu$, and (iii) an additional excitation appears with pure imaginary energy ν. By performing the quantization in the open string channel, we should be able to understand the meaning of these modes and reproduce the amplitude in the form of the partition function

$$\mathcal{Z}_{open} = -\frac{1}{2}\mathrm{Tr}\ln(L_0-1) = \frac{1}{2}\frac{dw}{w}\left(\frac{-1}{\ln w}\right)\mathrm{Tr}\, w^{L_0-1} . \quad (4.4)$$

4.2 Quantization, spectrum and the amplitude

Let us now perform the quantization in the open string channel [12]. We will employ the usual strip coordinate (σ, τ), where $0 \le \sigma \le \pi$. As in the calculation in the closed channel, we split the string coordinate into the classical part X_{cl}^μ and the quantum fluctuation part ξ^μ. In the present coordinate system, the classical solution takes the form $X_{cl}^\mu(\sigma,\tau) = \frac{\sigma}{\pi}(f_2^\mu(t_2) - f_1^\mu(t_1)) + f_1^\mu(t_1)$, which describes a string stretched from $f_1^\mu(t_1)$ to $f_2^\mu(t_2)$. Remembering that the components of $f_2 - f_1$ in the trajectory plane must vanish by consistency, the energy associated with this stretching is $H_{cl} = (f_1(t_1) - f_2(t_2))_\perp^2/(4\pi^2\alpha')$, which immediately gives the factor $w^{(f_1-f_2)_\perp^2/(4\pi^2\alpha')}$ appearing in (4.2).

As for the fluctuation part, we will continue to use the orthonormal frame defined in (2.36). Since the $D-2$ components ξ_I transverse to the trajectories satisfy the usual Dirichlet conditions and their quantization is standard, we will concentrate on the tangent components ξ_1 and ξ_2. By now the appropriate boundary conditions shoud be familiar:

$$\partial_\sigma \xi_1(0,\tau) = 0 , \quad \xi_2(0,\tau) = 0 , \quad (4.5)$$

$$\xi_1(\pi,\tau)\sinh\chi + \xi_2(\pi,\tau)\cosh\chi = 0 , \quad (4.6)$$

$$\partial_\sigma\xi_1(\pi,\tau)\cosh\chi + \partial_\sigma\xi_2(\pi,\tau)\sinh\chi = 0 . \quad (4.7)$$

$$(4.8)$$

[12] The calculation will turn out to be quite similar to the one perfomed in [8] (see also [12, 14]) but we will discuss the subtleties involved and their physical interpretation in much more detail.

It is not difficult to find the solutions of the free wave equation satisfying these conditions and expand ξ_1 and ξ_2 in terms of these modes. A convenient way of writing the expansions is

$$\begin{aligned}
\xi_1 &= \frac{\sqrt{\alpha'}}{\lambda_0^+}\left(e^{(\chi/\pi)\tau}\bar{\beta} + e^{-(\chi/\pi)\tau}\beta\right)\cosh\frac{\chi}{\pi}\sigma \\
&\quad +\sqrt{\alpha'}\sum_{n\geq 1,\epsilon=\pm}\frac{1}{\lambda_n^\epsilon}\left(\alpha_n^\epsilon e^{-i\lambda_n^\epsilon\tau} - \alpha_{-n}^\epsilon e^{i\lambda_n^\epsilon\tau}\right)\cos\lambda_n^\epsilon\sigma \qquad (4.9)\\
\xi_2 &= -\frac{\sqrt{\alpha'}}{\lambda_0^+}\left(e^{(\chi/\pi)\tau}\bar{\beta} + e^{-(\chi/\pi)\tau}\beta\right)\sinh\frac{\chi}{\pi}\sigma \qquad (4.10)\\
&\quad +i\sqrt{\alpha'}\sum_{n\geq 1,\epsilon=\pm}\frac{\epsilon}{\lambda_n^\epsilon}\left(\alpha_n^\epsilon e^{-i\lambda_n^\epsilon\tau} - \alpha_{-n}^\epsilon e^{i\lambda_n^\epsilon\tau}\right)\sin\lambda_n^\epsilon\sigma \qquad (4.11)
\end{aligned}$$

where $\lambda_m^\pm$ are given by

$$\lambda_m^\pm = m \pm i\frac{\chi}{\pi}\ . \qquad (4.12)$$

As was already anticipated from the modular transformed form (4.2), the eigenfrequencies are complex and the modes grow or diminish as functions of τ. Technically, this comes about because the boundary conditions involve hyperbolic functions of χ while the oscillatory part is trigonometric. Physical origin will be discussed after we finish the quantization.

Since the creation and the annihiliation operators should have opposite τ-dependence, one expects that the conjugate pairs are $(\alpha_n^+, \alpha_{-n}^+)$, $(\alpha_n^-, \alpha_{-n}^-)$, and $(\beta, \bar{\beta})$. Indeed it is easy to check that the canonical quantization conditions

$$\left[\dot{\xi}_i(\tau,\sigma), \xi_j(\tau,\sigma')\right] = \frac{2\pi\alpha'}{i}\delta(\sigma-\sigma')\eta_{ij} \qquad (4.13)$$

are satisfied if we impose the following commutation relations:

$$\left[\alpha_m^+, \alpha_{-n}^+\right] = \lambda_m^+\delta_{mn}\,, \quad \left[\alpha_m^-, \alpha_{-n}^-\right] = \lambda_m^-\delta_{mn}\,, \quad \left[\beta, \bar{\beta}\right] = \lambda_0^+\,. \qquad (4.14)$$

It is then a bit tedious but straightforward to compute the Virasoro operators L_n^{12} for the ξ_1-ξ_2 system by the expansion

$$\sum_n L_n^{12}e^{-in(\tau+\sigma)} = \frac{1}{4\alpha'}\left[-(\partial_\tau\xi_1 + \partial_\sigma\xi_1)^2 + (\partial_\tau\xi_2 + \partial_\sigma\xi_2)^2\right]\,. \qquad (4.15)$$

(One can check that on the right hand side terms which do not have the dependence $e^{-in(\tau+\sigma)}$ all cancel.) The result is, with $n \geq 1$,

$$L_0^{12} = -\bar{\beta}\beta + \sum_{m\geq 1}(\alpha_{-m}^+\alpha_m^+ + \alpha_{-m}^-\alpha_m^-) + \frac{1}{2}\nu(1-\nu)\,, \qquad (4.16)$$

$$
\begin{aligned}
L_n^{12} &= \bar{\beta}\alpha_n^- - \beta\alpha_n^+ + \sum_{l+m=n;l,m\geq 1} \alpha_l^+\alpha_m^- \\
&\quad + \sum_{m\geq 1} \left(\alpha_{-m}^+\alpha_{m+n}^+ + \alpha_{-m}^-\alpha_{m+n}^-\right) , \qquad (4.17) \\
L_{-n}^{12} &= \bar{\beta}\alpha_{-n}^+ - \beta\alpha_{-n}^- + \sum_{l+m=n;l,m\geq 1} \alpha_{-l}^+\alpha_{-m}^- \\
&\quad + \sum_{m\geq 1, m-n\geq 1} \left(\alpha_{-m}^+\alpha_{m-n}^+ + \alpha_{-m}^-\alpha_{m-n}^-\right) . \qquad (4.18)
\end{aligned}
$$

The shift $\nu(1-\nu)/2$ in L_0^{12}, which takes exactly the form expected, has been determined so that these operators satisfy the usual form of the Virasoro algebra (with central charge equal to 2):

$$
\left[L_m^{12}, L_n^{12}\right] = (m-n)L_{m+n}^{12} + \delta_{m+n,0}\frac{2}{12}(m^3-m) . \qquad (4.19)
$$

Also note that L_0^{12} is hermitian with this shift included.

Now, since L_0^{12} appears to be diagonal in the number operators for the oscillator modes, one may expect that the trace over the modes, $\mathrm{Tr}w^{L_0^{12}}$, readily leads to the expressions in (4.2). This naive reasoning must however be carefully examined because of the unusual hermiticity properties of the oscillators. From the explicit form of ξ_1 and ξ_2 given in (4.9) and (4.11), one finds that the reality of these fields requires

$$
\begin{aligned}
(\alpha_m^+)^\dagger &= -\alpha_{-m}^- , \qquad (\alpha_m^-)^\dagger = -\alpha_{-m}^+ , \qquad (4.20) \\
\beta^\dagger &= \beta , \qquad \bar{\beta}^\dagger = \bar{\beta} . \qquad (4.21)
\end{aligned}
$$

This shows that, while β and $\bar{\beta}$ are like a coordinate and its conjugate momentum, $\alpha_n^\pm$ are interchanged under hermitian conjugation and hence cannot be identified as usual oscillators.

To study the structure of the Hilbert space for these unusual $\alpha_n^\pm$ oscillators, let us concentrate on a particular level n and consider the system defined by

$$
\begin{aligned}
[a, \tilde{a}] &= \lambda , \qquad \left[b, \tilde{b}\right] = \lambda^* , \qquad (4.22) \\
a^\dagger &= -\tilde{b} , \qquad \tilde{a}^\dagger = -b , \qquad (4.23) \\
H &= \tilde{a}a + \tilde{b}b , \qquad (4.24)
\end{aligned}
$$

where λ is a non-vanishing complex number. The Hamiltonian H is obviously hermitian and the system is completely consistent. Suppose we define the vacuum state $\psi_{0,0}$ by

$$
a\psi_{0,0} = 0 , \qquad b\psi_{0,0} = 0 . \qquad (4.25)
$$

Then we can build a general excited state $\psi_{m,n}$ by

$$\psi_{m,n} \;=\; N_{m,n}\tilde{a}^m\tilde{b}^n\psi_{0,0}\,, \tag{4.26}$$

where $N_{m,n}$ is a normalization constant. This is clearly an eigenstate of H with the eigenvalue $m\lambda + n\lambda^*$, which is *complex* unless $m = n$. It is well known that a hermitian operator can have complex eigenvalues if and only if the norm of the corresponding eigenstates vanish [18]. Indeed we have

$$\begin{aligned}(\psi_{m,n}, \tilde{a}a\psi_{m,n}) &= m\lambda(\psi_{m,n}, \psi_{m,n}) \\ &= (\tilde{b}b\psi_{m,n}, \psi_{m,n}) = n\lambda(\psi_{m,n}, \psi_{m,n})\,,\end{aligned} \tag{4.27}$$

and the norm $(\psi_{m,n}, \psi_{m,n})$ vanishes for $m \neq n$. By a similar manipulation, one can easily show that the state which has non-vanishing inner product with $\psi_{m,n}$ is $\psi_{n,m}$. In other words, the metric of this Hilbert space is non-diagonal[13]. If we choose the normalization so that $(\psi_{n,m}, \psi_{m,n}) = 1$, then the trace of an operator $\mathcal{O}$ should be defined as

$$\mathrm{Tr}\,\mathcal{O} \;\equiv\; \sum_{m\geq 0, n\geq 0} (\psi_{n,m}, \mathcal{O}\psi_{m,n})\,. \tag{4.28}$$

Hence the partition function is computed as

$$\begin{aligned}\mathrm{Tr}\,w^H &= \sum_{m\geq 0, n\geq 0} (\psi_{n,m}, w^H\psi_{m,n}) \\ &= \sum_{m\geq 0, n\geq 0} w^{m\lambda+n\lambda^*} = \left(\sum_{m=0}^{\infty} w^{m\lambda}\right)\left(\sum_{n=0}^{\infty} w^{m\lambda^*}\right) \\ &= (1-w^{\lambda})^{-1}(1-w^{\lambda^*})^{-1}\,,\end{aligned} \tag{4.29}$$

which turned out to be identical with the result expected by the naive reasoning.

To understand the physical meaning of this structure as well as to check the result in a different way, it is instructive to form new oscillators which satisfy the conventional hermiticity property. For example, if we define $A, A^\dagger, B, B^\dagger$ as

$$A \;=\; \frac{1}{\sqrt{2l|\lambda|}}(\lambda a + \lambda b) \tag{4.30}$$

$$A^\dagger \;=\; \frac{-1}{\sqrt{2l|\lambda|}}(\lambda^*\tilde{a} + \lambda^*\tilde{b}) \tag{4.31}$$

$$B \;=\; \frac{1}{\sqrt{2l|\lambda|}}(\lambda^* a - \lambda b) \tag{4.32}$$

$$B^\dagger \;=\; \frac{1}{\sqrt{2l|\lambda|}}(\lambda^*\tilde{a} - \lambda\tilde{b})\,, \tag{4.33}$$

[13]This means that the Hilbert space has negative norm states, but this is simply due to the timelike nature of ξ_1.

where l is the real part of λ and is taken positive, they satisfy the standard commutation relations $\left[A, A^\dagger\right] = -1\,, \left[B, B^\dagger\right] = 1$. In terms of these oscillators, the Hamiltonian becomes

$$
\begin{aligned}
H &= \tilde{a}a + \tilde{b}b \\
&= \frac{|\lambda|^2}{l}\left\{B^\dagger B - \frac{1}{2}\left(\frac{\lambda}{\lambda^*} + \frac{\lambda^*}{\lambda}\right) A^\dagger A \right. \\
&\quad \left. +\frac{1}{2}\left(1 - \frac{\lambda}{\lambda^*}\right) A^\dagger B + \frac{1}{2}\left(1 - \frac{\lambda^*}{\lambda}\right) B^\dagger A\right\}. \qquad (4.34)
\end{aligned}
$$

Thus, in this representation the metric of the Hilbert space is diagonal but instead the Hamiltonian is non-diagonal for complex λ. As one can easily check, A and B are, respectively, the oscillators which describe the usual particle modes for ξ_1 and ξ_2 when λ becomes real. This means that, except for a special situation where χ vanishes, the usual particle-like modes are "unstable" and as time goes on they incessantly transform into each other. Nevertheless, since the Hamiltonian is hermitian, the total probability is conserved and the process is unitary.

As for the calculation of the trace, although a bit tedious, one can reproduce the previous result (4.29) using the above non-diagonal Hamiltonian (for example by the use of the coherent state method). Finally, applying this to the original problem, we find that the contribution of the $\alpha^\pm_m, (m \neq 0)$ oscillators to the trace $\mathrm{Tr}\, w^{L_0^{12}}$ is given by

$$
\prod_{m=1}^{\infty} (1 - w^{m+\nu})^{-1} (1 - w^{m-\nu})^{-1}\,, \qquad (4.35)
$$

as anticipated.

Let us next consider the β-$\bar{\beta}$ system, which is characterized by

$$
\left[\beta, \bar{\beta}\right] = i\frac{\chi}{\pi}\,, \qquad (4.36)
$$

$$
\beta^\dagger = \beta\,, \qquad \bar{\beta}^\dagger = \bar{\beta}\,, \qquad (4.37)
$$

$$
H = -\bar{\beta}\beta - \frac{i\chi}{2\pi} = H^\dagger\,. \qquad (4.38)
$$

Before performing a proper computation of the trace $\mathrm{Tr}\, w^H = \mathrm{Tr}\, e^{-\pi t H}\,, t = -\ln w/\pi$ for this system, let us describe the essence of the physics by looking at the small χ limit. In this limit, the boundary conditions for ξ_1 and ξ_2 become almost Neumann and Dirichlet respectively. Indeed if we introduce q and p by

$$
q = \frac{\sqrt{\alpha'}}{i}(\beta + \bar{\beta})\,, \qquad p = \frac{i}{2\sqrt{\alpha'}}(\beta - \bar{\beta})\,, \qquad (4.39)
$$

$$
[p, q] = i\frac{\chi}{\pi}\,, \qquad (4.40)
$$

ξ_i can be written in the form

$$\begin{aligned}
\xi_1 &= \frac{\pi q}{\chi} + 2\alpha' p\tau + \mathcal{O}(\chi^2)\,, && (4.41)\\
\xi_2 &= -q\sigma + \mathcal{O}(\chi^2)\,, && (4.42)
\end{aligned}$$

with the energy

$$H = \frac{1}{4\alpha'}q^2 - \alpha' p^2\,. \qquad (4.43)$$

This corresponds to the following picture: When χ is small, while the $\sigma = 0$ end of the string can fluctuate strictly along the u_1 direction, the $\sigma = \pi$ end may slide along the direction which is slightly tilted from u_1 into the perpendicular $\tilde{u}_2$ direction. Thus, the string, as a whole without oscillation, can move into the u_1 direction almost freely but this motion is accompanied by a stretching or shrinking in the $\tilde{u}_2$ direction. This costs energy and results in the restoring potential seen in the Hamiltonian. Now from (4.40) we see that as χ tends to vanish, p and q will become independent. Therefore, in this limit we get

$$\mathrm{Tr}\, e^{-\pi t H} \quad\longrightarrow\quad \frac{\pi}{\chi}\int dq dp e^{-\pi t q^2/4\alpha'} e^{\pi t\alpha' p^2} \sim \frac{1}{\chi \ln w}\,, \qquad (4.44)$$

where the factor in front of the integral compensates the normalization of the commutator (4.40). One can easily check that this agrees with the $\chi \to 0$ limit of the expression $1/(1-w^\nu)$ in (4.2).

Let us now compute $\mathrm{Tr}\, e^{-\pi t H}$ for general χ. In order to build a well-defined Hilbert space, we shall define a conventional oscillator pair $(a, a^\dagger)$ by linear combinations of β and $\bar{\beta}$. A simple choice that realizes $\left[a, a^\dagger\right] = 1$, $(a)^\dagger = a^\dagger$ is

$$a = \frac{1}{\sqrt{2\chi/\pi}}(\beta + i\bar{\beta})\,, \qquad a^\dagger = \frac{1}{\sqrt{2\chi/\pi}}(\beta - i\bar{\beta})\,. \qquad (4.45)$$

(More general ones are unitarily equivalent to this.) Then the Hamiltonian takes the form

$$H = \frac{i\chi}{2\pi}(a^2 - {a^\dagger}^2 + 1)\,, \qquad (4.46)$$

which is not number-diagonal. The relevant trace $\mathrm{Tr}\, e^{-\pi t H}$ can nevertheless be computed in various ways. One way is to first perform the normal ordering by a technique similar to the one explained in the Appendix, which gives

$$
\begin{aligned}
e^{-\pi t H} &= \delta e^{\alpha a^{\dagger 2}} e^{\beta a^2} e^{\gamma a^\dagger a}\,, && (4.47)\\
\text{where} \qquad \delta &= e^{-i\chi t/2}(\cos\chi t)^{-1/2}\,, && (4.48)\\
\alpha &= \frac{i}{2}\tan\chi t\,, && (4.49)\\
\beta &= -\frac{i}{2}\sin\chi t\cos\chi t\,, && (4.50)\\
\gamma &= -\ln(\cos\chi t)\,. && (4.51)
\end{aligned}
$$

Then the trace of this expression is easily obtained by using the coherent state method. The result is

$$
\begin{aligned}
\mathrm{Tr}\, e^{-\pi t H} &= \delta\left[(1-e^\gamma)^2 - 4\alpha\beta e^{2\gamma}\right]^{-1/2} && (4.52)\\
&= \frac{e^{-i\chi t/2}}{2i\sin\frac{\chi t}{2}} = \frac{1}{1-w^\nu}\,. && (4.53)
\end{aligned}
$$

Thus we obtain the desired expression.

We now briefly describe the contributions of the $D-2$ transverse components and of the ghosts.

As for ξ_I's, which satisfy the standard Dirichlet boundary conditions, there are no zero modes and their contribution therefore is simply $f(w)^{-(D-2)}$.

Finally the ghosts. Since they are not affected by the background, the analysis is the same as in the usual case, which was described in detail in the appendix of [12]. The essential point is the following. Recall that due to the presence on a cylinder of a Teichmüller parameter and a conformal symmetry the calculation using the boundary states in the closed string channel required an insertion of appropriate zero modes, which is equivalent to an insertion of the ghost number operator. When converted into the open string channel, the effect of this insertion produces a factor of $\ln w$. In this way, one obtains the formula connecting the ghost partition function in the closed channel and that in the open channel:

$$
q^{1/6} f(q^2)^2 = w^{1/12} f(w)^2(-\ln w/2\pi)\,. \qquad (4.54)
$$

This, however, is nothing but the famous modular transformation formula for the (square of) the Dedekind η function. Since the powers of q and w appearing in this formula have already been taken care of by the total intercept of the Virasoro operator, the contribution we must add is $f(w)^2(-\ln w/2\pi)$. In particular, this factor of $\ln w$ is exactly what is needed to cancel the $1/\ln w$ in (4.4).

Thus, putting everything together, we have reproduced all the factors that appear in the scattering amplitude as seen in the open channel. The important lesson we learned from this exercise is that non-trivial mixing of particle modes occur in this channel. We are intending to understand this phenomenon from the point of view of effective gauge theory in a future report.

5 Discussions

In this article, we have computed the amplitude for the scattering of two D-particles in bosonic string theory, where D-particles themselves are quantized. We employed three different methods, namely, the path integral, the boundary state, and the operator formalism in the open string channel, and cross-checked the result. The emphasis was on the path integral method, which is conceptually most complete in formulating the problem. Especially, it is only with this method that we can clearly grasp the nature of the approximation used and compute the corrections in a systematic manner. As far as the computation in the lowest order (in the acceleration $\ddot{f}_i$) is concerned, the use of the boundary state representation of the interaction vertex appears to be most efficient. On the other hand, the operator method in the open string channel reveals the occurence of incessant transitions among the excitations of the open string that connect the D-particles and perhaps a deeper understanding of this behavior will be important in unravelling the connection with the description in terms of the spontaneously broken gauge theory.

In any event, the fact that the interaction between *quantum* D-particles through all the excitation modes of a string is fully consistent is rather remarkable. This is because in the present approach D-particles are no longer backgrounds but are new independent entities which can coexist with strings. This raises a puzzling question[14]: Is it not true that they are supposed to be solitons of string theory and hence they should not represent new degrees of freedom ? This may indicate that some important non-perturbative consistency condition linking the two is still missing. On the other hand, the conjecture[19] that the D-particles are the Kaluza-Klein modes of 11 dimensional M theory indicates that they could indeed be degrees of freedom independent of those of strings. The answer is not yet known, but it is certainly an important question in the light of consistency of our result.

In this article, we have presented the calculation only for bosonic string theory and have left out the more important case of superstring. The reason is that in the latter case, a satisfactory analysis requires the understanding of not only the scattering of bosonic

[14]This question was raised by N. Ishibashi during our discussion.

D-particles but also that involving fermionic superpartners as well and this appears to be non-trivial. In the path integral approach, we need to consider supertrajectories and their quantization and this is known to be difficult. This matter is under investigation and we hope to be able to report our progress in the near future.

Nevertheless, if one is content with the amplitude for two bosonic quantum D-particles scattering into again two bosonic ones, we can present a well-educated guess. This is due to the fact that in the bosonic string case (at least for the lowest order approximation) there emerged a simple rule to go from the forward scattering amplitude for infinitely heavy D-particles to our general amplitude for quantized D-particles with finite mass and it is almost obvious that this rule should continue to hold for the superstring case. Assuming this to be the case, all we have to do is to write down the result obtained in [8], make one simple substitution and a Fourier transformation, and take into account the quantum indistinguishability discussed in subsection 2.4. Explicitly, the general on-shell amplitude in the open channel representation should read

$$\mathcal{A}_{super}(p_i, p_i') = \mathcal{A}_{super}(p_1, p_2; p_1', p_2') + \mathcal{A}_{super}(p_1, p_2; p_2', p_1')\,, \tag{5.1}$$

$$\tag{5.2}$$

where

$$\mathcal{A}_{super}(p_1, p_2; p_1', p_2') \sim \int_0^\infty \frac{dt}{t} t^{-4} e^{-\alpha' \pi (p_1'-p_1)^2/t} \frac{\theta_1'\left(0|\frac{it}{2}\right)}{\theta_1\left(\frac{\chi t}{2\pi}|\frac{it}{2}\right)} \times \sum_{\alpha=2,3,4} \theta_\alpha\left(\frac{\chi t}{2\pi}|\frac{it}{2}\right)\theta_\alpha^3\left(0|\frac{it}{2}\right)\eta^{-12}\left(\frac{it}{2}\right)\,. \tag{5.3}$$

In this formula, χ is as given in (2.140) $\sim$ (2.141) and $e_2 = -e_3 = e_4 = -1$. We intend to discuss in detail how this general formula fits with the super-Yang-Mills description in a future communication.

We finish by making an important observation that the type of amplitude for the basic process considered in this paper, no matter how accurately computed in superstring theory and no matter how small the string coupling constant is, cannot be relied upon if one wishes to know the behavior of the D-particles at very high energy. The difficulty stems from the fact that the annulus amplitude does not explicitly depend on the string coupling constant g_s. Thus, one can insert "annulus interactions" anywhere one likes and produce infinitely many diagrams, such as the ones dipicted in Fig.4, which contribute at the same order in g_s.

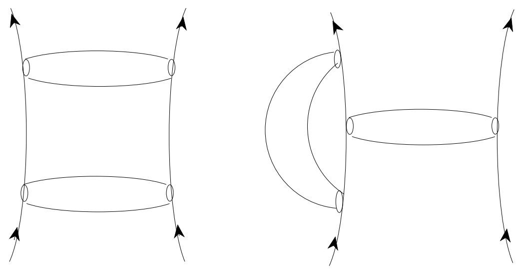

Fig.4 Example of diagrams at the same order in the string coupling as the basic one.

At energy much smaller than the D-particle mass, the diagrams with extra corrections are expected to be suppressed by powers of E^2/m^2 due to the presence of internal propagators of D-particles. But as the energy becomes comparable to m all these diagrams would contribute equally to the amplitude and computation appears to become intractable. It would however be extremely interesting if one can devise a method to extract the leading high energy behavior out of these diagrams just as the similar study for ordinary strings [20] stimulated a lot of thinking about the degrees of freedom of string theory at short distance.

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Appendix: Normal Ordering Formula

Let $(a_i, a_i^\dagger)$ and $(\tilde{a}_i, \tilde{a}_i^\dagger)$ be two independet sets of oscillators satisfying the usual (anti-)commutation relations $\left[a_i, a_j^\dagger\right]_\pm = \left[\tilde{a}_i, \tilde{a}_j^\dagger\right]_\pm = \delta_{ij}$. Throughout, upper (lower) sign refers to the anti-commuting (commuting) case. Consider the bilinears

$$\alpha = aA\tilde{a}\,, \qquad \beta = \tilde{a}^\dagger B a^\dagger\,, \tag{A.1}$$

where $aA\tilde{a} \equiv \sum_{i,j} a_i A_{ij} \tilde{a}_j$, etc.. By extending the idea of [17], we will derive a formula which gives the fully normal ordered form of $e^\alpha e^\beta$.

Let λ be a parameter and we seek the result in the following form:

$$\begin{aligned} e^{\lambda\alpha}e^{\lambda\beta} &= \delta(\lambda)e^{x(\lambda)}e^{y(\lambda)}e^{z(\lambda)}\,, & \text{(A.2)} \\ x(\lambda) &= \tilde{a}^\dagger X(\lambda) a^\dagger\,, & \text{(A.3)} \\ y(\lambda) &= a Y(\lambda)\tilde{a}\,, & \text{(A.4)} \\ z(\lambda) &= \tilde{a}^\dagger Z(\lambda)\tilde{a} + a^\dagger W(\lambda) a\,. & \text{(A.5)} \end{aligned}$$

Here $X(\lambda), Y(\lambda), Z(\lambda), W(\lambda)$ are matrices and $\delta(\lambda)$ is a prefactor, to be determined. Let us define

$$\begin{aligned} \chi_1(\lambda) &= \delta(\lambda)e^{x(\lambda)}e^{y(\lambda)}\,, & \text{(A.6)} \\ \chi_2(\lambda) &= e^{\lambda\alpha}e^{\lambda\beta}e^{-z(\lambda)}\,. & \text{(A.7)} \end{aligned}$$

Then what we want to obtain is $\chi_1(\lambda) = \chi_2(\lambda)$ and this is achieved uniquely if we can satisfy

$$\begin{aligned} &(i) \quad \chi_1(0) = \chi_2(0)\,, & \text{(A.8)} \\ &(ii) \quad \chi_1^{-1}(\lambda)\chi_1'(\lambda) = \chi_2^{-1}(\lambda)\chi_2'(\lambda)\,, & \text{(A.9)} \end{aligned}$$

where $\chi_1' \equiv d\chi_1/d\lambda$ etc.. Writing out the condition (ii) explicitly, we get

$$\frac{\delta'}{\delta} + \chi_1^{-1}x'(\lambda)\chi_1 + y'(\lambda) = \chi_2^{-1}\alpha\chi_2 + \chi_2^{-1}e^{\lambda\alpha}\beta e^{-\lambda\alpha}\chi_2 - z'(\lambda)\,, \qquad \text{(A.10)}$$

where we have made an assumption $[z(\lambda), z'(\lambda)] = 0$. Its validity will be aposteriori justified. Evaluation of the quantities appearing in (A.10) is a bit tedious but straightforward. Then, by equating the coefficients of the same operator structure, we get the following 5 differential equations:

$$\begin{aligned} &(a) \quad \frac{\delta'}{\delta} = \lambda \mathrm{Tr}(BA) + \mathrm{Tr}(X'Y)\,, & \text{(A.11)} \\ &(b) \quad X' = e^Z\left(1 \mp \lambda^2 BA\right)Be^{W^T}\,, & \text{(A.12)} \\ &(c) \quad Y' \mp YX'Y = e^{-W^T}Ae^{-Z}\,, & \text{(A.13)} \\ &(d) \quad X'Y = \mp Z' - \lambda e^Z BAe^{-Z}\,, & \text{(A.14)} \\ &(e) \quad YX' = W'^T - \lambda e^{-W^T}ABe^{W^T}\,. & \text{(A.15)} \end{aligned}$$

From the structure of these equations, it is consistent to assume that $X'Y, Z, X'A, BY$ are functions only of BA, while YX' and W^T depend only on AB. Let us take the trace of (d) and (e). We get

$$\begin{aligned} \mathrm{Tr}(X'Y) &= \mathrm{Tr}Z' - \lambda\mathrm{Tr}(BA) \\ &= \mathrm{Tr}W' - \lambda\mathrm{Tr}(BA)\,. & \text{(A.16)} \end{aligned}$$

Thus we must have $\mathrm{Tr} Z' = \mathrm{Tr} W'$. Substituting (A.16) into (a) and integrating, we get

$$\delta \;=\; e^{\mathrm{Tr} Z} = e^{\mathrm{Tr} W} \,. \tag{A.17}$$

This result, together with the previously mentioned dependence on AB and BA, prompts us to make a postulate

$$Z \;=\; f(BA) \,, \qquad W^T = f(AB) \,, \tag{A.18}$$

where $f(x)$ is some power-expandable function. Then since $Bf(AB) = f(BA)B$ holds, we have

$$BW^T \;=\; ZB \,, \qquad W^T A = AZ \,. \tag{A.19}$$

Using these postulates, the equations (b) $\sim$ (d) simplify to

$$(b) \quad X' = e^{2Z} \left(1 \mp \lambda^2 \xi\right) B \,, \tag{A.20}$$

$$(c) \quad Y' \mp Y X' Y = A e^{-2Z} \,, \tag{A.21}$$

$$(d) \quad X' Y = \mp Z' - \lambda \xi \,, \tag{A.22}$$

$$\text{where} \qquad \xi \equiv BA \,. \tag{A.23}$$

To solve them, first multiply (b) from right by Y. We get

$$X' Y \;=\; e^{2Z} \left(1 \mp \lambda^2 \xi\right) \eta \,, \tag{A.24}$$

$$\text{where} \qquad \eta \;\equiv\; BY \,. \tag{A.25}$$

Equating this with the RHS of (d), we can solve for η in terms of ξ:

$$\eta \;=\; (1 \mp \lambda^2 \xi)^{-1} e^{-2Z} (\mp Z' - \lambda \xi) \,. \tag{A.26}$$

Next multiply (c) from left by B and substitute (d) for $X'Y$. One then obtains

$$\eta' \mp \eta (\mp Z' - \lambda \xi) \;=\; \xi e^{-2Z} \,. \tag{A.27}$$

We may now eliminate η by substituting (A.26). After some calculation we get a differential equation for Z of the form

$$(1 \mp \lambda^2 \xi)(Z'^2 - Z'') \;=\; \pm 2 \lambda \xi Z' \pm 2\xi \,. \tag{A.28}$$

The solution of this equation satisfying the proper boundary condition $Z(0) = 0$ is given by

$$Z \;=\; -\ln(1 \pm \lambda^2 \xi) \,. \tag{A.29}$$

We then get from (A.17) and the postulate (A.18)

$$\delta = \exp\left(\pm\frac{1}{2}\mathrm{Tr}\ln(1\pm\lambda^2\xi)\right) = \left[\det(1\pm\lambda^2\xi)\right]^{\pm 1/2}, \qquad (A.30)$$

$$W = f(B^TA^T) = -\ln\left(1\pm\lambda^2 B^TA^T\right). \qquad (A.31)$$

The matrix X, satisfying $X(0)=0$, is obtained by integrating (b). We get

$$X = \int_0^\lambda du \frac{1\mp u^2\xi}{(1\pm u^2\xi)^2}B = \frac{\lambda}{1\pm\lambda^2\xi}B. \qquad (A.32)$$

To get Y, we recall $Y = B^{-1}\eta$, where η is given by (A.26). One immediately gets

$$Y = \lambda A(1\pm\lambda^2\xi). \qquad (A.33)$$

The remaining equation (e), which now reads

$$YX' = \mp W'^T - \lambda AB, \qquad (A.34)$$

is automatically satisfied. Also one can easily check the validity of the assumption made earlier, namely $[z(\lambda), z'(\lambda)] = 0$. Since the solution of the system of equations with appropriate boundary conditions is unique, we can justify all the postulates made above.

Putting all together, we get the normal-ordering formula quoted in the text:

$$\begin{aligned} & e^{aA\tilde{a}}e^{\tilde{a}^\dagger Ba^\dagger} \\ & \quad = [\det(1\pm BA)]^{\pm 1}\, e^{\tilde{a}^\dagger(1\pm BA)^{-1}Ba^\dagger} \\ & \quad \cdot e^{aA(1\pm BA)\tilde{a}}e^{-\tilde{a}^\dagger\ln(1\pm BA)\tilde{a}}e^{-a^\dagger\ln(1\pm B^TA^T)a}. \end{aligned} \qquad (A.35)$$

For completeness, we exhibit the formula for the case of one set of oscillators, which can be obtained in a similar manner:

$$\begin{aligned} & e^{\frac{1}{2}aAa}e^{\frac{1}{2}a^\dagger Ba^\dagger} \\ & \quad = [\det(1\pm BA)]^{\pm 1/2}\, e^{\frac{1}{2}a^\dagger(1\pm BA)^{-1}Ba^\dagger} \\ & \quad \cdot e^{\frac{1}{2}aA(1\pm BA)a}e^{-a^\dagger\ln(1\pm BA)a}. \end{aligned} \qquad (A.36)$$

In this case, the matrix A and B are symmetric (anti-symmetric) if the oscillators are commuting (anti-commuting).

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